Population extinction times and consequences of an incorrect model specification

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Summary

- We use stochastic differential equations (SDE) to model population growth in a randomly varying environment.
- We start with models based on classical deterministic models, such as the logistic or the Gompertz model, taken as approximate models of the "true" (usually unknown) average growth rate.
- Can we trust predictions based on the classical (simpler but approximate) models within a certain degree of accuracy?
- We study the effect of the gap between the approximate and the "true" model, on model predictions, particularly on asymptotic behavior and mean and variance of the population time to extinction.

Deterministic growth models

$$\frac{1}{X(t)}\frac{dX(t)}{dt} = f(X(t)) \qquad X(0) = x > 0 \quad \text{is known}$$

where

$X(t)$ population size at time t

$f(x)$ (per capita) growth rate when $X=x$ (defined for $z>0$)

Classical approximate models	Logistic	$f(x) = r\,(1 - x/K)$
	Gompertz	$f(x) = r\ \ln(K/x)$

$r>0$ "intrinsic" growth rate
$K>0$ carrying capacity of the environment

Randomly fluctuating environment

Effect of environmental random fluctuations

$\varepsilon(t)$ standard white noise $\quad W(t) = \int_0^t \varepsilon(s)ds$ Wiener process

$\sigma > 0$ noise intensity parameter

SDE approximate models: Autonomous stochastic differential equation (SDE)

$$\frac{1}{X}\frac{dX(t)}{dt} = f(X) + \sigma\varepsilon(t) \quad \text{or} \quad dX(t) = f(X(t))X(t)dt + \sigma X(t)dW(t)$$

with Logistic $f(x) = r\,(1 - x/K)$

or Gompertz $f(x) = r\,\ln(K/x)$

SDE "true" models

$$dX(t) = Xf(X)\ dt + \sigma X\ dW(t)$$

with $f(x) = r\,(1 - x/K) + \alpha(x)$

or $f(x) = r\,\ln(K/x) + \alpha(x)$

α is a C^1 function s.t. $|\alpha(x)| / r < \delta$

$$dX(t) = Xf(X)\ dt + (\sigma + \alpha(X))\ X\ dW(t)$$

with $f(x) = r\,(1 - x/K)$

or $f(x) = r\,\ln(K/x)$

α is a C^2 function s.t. $|\alpha(x)| / \sigma < \delta$

Randomly fluctuating environment

We will use Stratonovich calculus.

The solution X(t) is a homogeneous ergodic diffusion process with

drift coefficient (infinitesimal mean)

$a(x) = x\,\big(r\,(1 - x/K) + \boldsymbol{\alpha(x)} + \sigma^2/2\,\big)$

$a(x) = x\,\big(r\,(1 - x/K) + (1/2)(\sigma + \alpha(x))\,(\sigma + \alpha(x) + x\alpha'(x))\,\big)$

$a(x) = x\,\big(r\ln(K/x) + \boldsymbol{\alpha(x)} + \sigma^2/2\,\big)$

$a(x) = x\,\big(r\ln(K/x) + (1/2)(\sigma + \alpha(x))\,(\sigma + \alpha(x) + x\alpha'(x))\,\big)$

diffusion coefficient (infinitesimal variance)

$b^2(x) = \sigma^2 x^2$

$b^2(x) = (\sigma + \boldsymbol{\alpha(x)})^2 x^2$

In our case, a and b are C^1 functions, so the solution exists and is unique up to an explosion time.

Scale and speed measures

Scale meaure $S(a,b)=S(b)-S(a)$

Scale function $S(y)=\int_{y^{**}}^{y} s(z)dz$

Scale density $$s(z):=\exp\left(-\int_{y^*}^{z}\frac{2a(\theta)}{b^2(\theta)}d\theta\right)$$

Speed measure $M(a,b)=M(b)-M(a)$

Speed function $M(y)=\int_{x_{**}}^{y} m(z)dz$

Speed density $$m(z):=\frac{1}{s(z)b^2(z)}\propto \text{stationary density}$$

Diffusion operator $$\mathcal{D}=a(x)\frac{d}{dx}+\frac{1}{2}b^2(x)\frac{d^2}{dx^2}=\frac{1}{2}\frac{d}{dM(x)}\left(\frac{d}{dS(x)}\right)$$

Randomly fluctuating environment

- The state space has boundaries X=0 and X=+∞.
- On can see that X=0 and X=+∞ are non-attracting.
- There is no t such that $X(t)$=0 and so “mathematical” extinction has zero probability of occurring. “Realistic” extinction will, however, occur with probability one.
- Explosions can not occur and the solution exists and is unique for all t>0.
- There exists a stationary density $p(y)$, because

$$M = \int_0^{+\infty} m(z)dz < +\infty.$$

First passage times

- Let $a < x < b$.
- The first passage time of the population size $X(t)$ by a us denote by T_a. Similarly define T_b.

$$T_a = \inf\{t > 0 : X(t) = a\} \qquad T_b = \inf\{t > 0 : X(t) = b\}$$

$$T_{ab} = \min(T_a, T_b)$$

- The probability of $X(t)$ to reach a before reaching b is

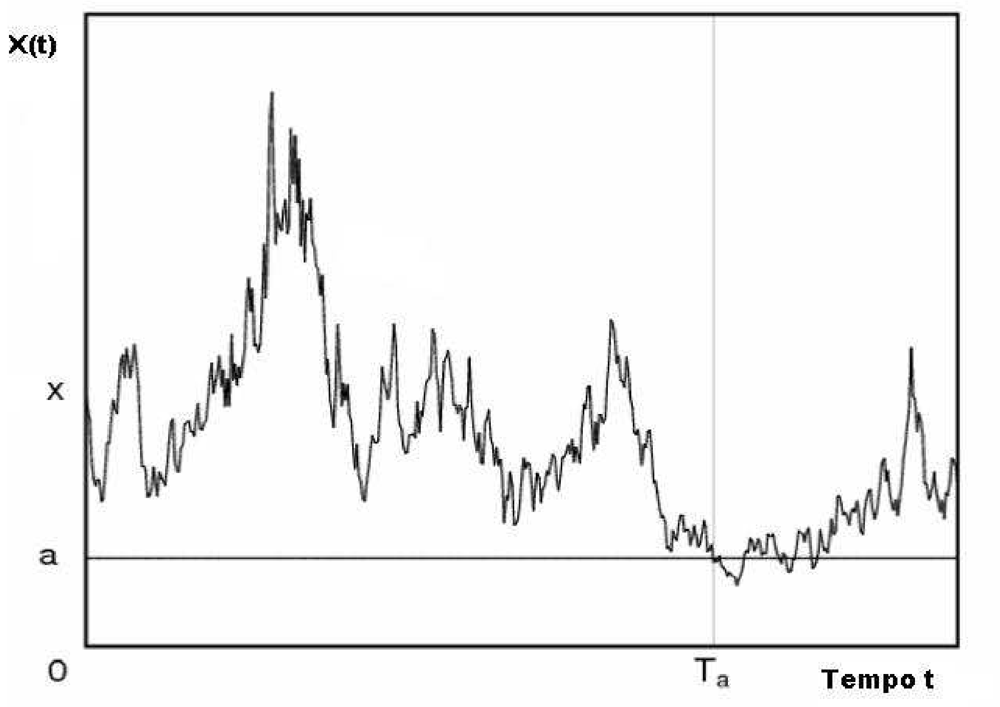
$$u(x) = \mathbf{P}\left[T_a < T_b \middle| X(0) = x\right] = \frac{S(x,b)}{S(a,b)}$$

Time to extinction

We are interested in the time required for the population to reach the extinction threshold a for the first time. Let us denote this extinction time by

$$T_a = \inf\{t > 0 : \; X(t) = a\}$$

$$a < x < +\infty$$



Time to extinction

$$U_k(x) = \mathbf{E}\left[(T_{ab})^k \middle| X(0) = x\right] \qquad k\text{ - th order moment}$$

$$\mathscr{D}\, U_k(x) + kU_{k-1}(x) = 0$$

$$\frac{1}{2}\frac{d}{dM(x)}\left(\frac{dU_k(x)}{dS(x)}\right) + kU_{k-1}(x) = 0$$

$$U_k(a) = U_k(b) = 0 \quad (k = 1,2,\ldots) \qquad U_0(x) \equiv 1$$

$$U_k(x) = 2u(x)\int_x^b S(\xi,b)\,kU_{k-1}(\xi)m(\xi)d\xi + 2(1-u(x))\int_a^x S(a,\xi)\,kU_{k-1}(\xi)m(\xi)d\xi$$

Since the process is ergodic, if we let $b \uparrow +\infty$, we obtain as limit of U_k(x)

$$V_k(x) = \mathbf{E}\left[(T_a)^k \middle| X(0) = x\right] = 2\int_a^x s(\xi)\left(\int_\xi^{+\infty} kV_{k-1}(\theta)m(\theta)d\theta\right)d\xi$$

Time to extinction

For k=1

$$\mathbf{E}\left[T_a \middle| X(0)=x\right] = V_1(a) = 2\int_a^x s(\xi)\left(\int_\xi^{+\infty} m(\theta)d\theta\right)d\xi$$

Replacing in the equation for k=2 we obtain $V_2(x)$ and then

$$\mathbf{VAR}\left[T_a \middle| X(0)=x\right] = \int_a^x s(\xi)\int_\xi^{+\infty} s(\mu)\left(\int_\mu^{+\infty} m(\theta)d\theta\right)^2 d\mu d\xi$$

These have to be numerically integrated.

First passage times

We can also obtain an ODE for the Laplace transform

$$U_\lambda(x) = \mathbf{E}\left[\exp(-\lambda T_{ab}) \middle| < X(0) = x\right]$$

$$\frac{1}{2}\frac{d}{dM(x)}\left(\frac{dU_\lambda(x)}{dS(x)}\right) - \lambda U_\lambda(x) = 0$$

$$U_\lambda(a) = U_\lambda(b) = 1$$

Solving the equation and inverting the Laplace transform, one obtains the p.d.f. of T_{ab}

Logistic model, approximate drift coefficient

$$|\beta(X)| = \frac{|\alpha(X)|}{r} \le \delta, \quad R = \frac{r}{\sigma^2}, \quad d = \frac{a}{K}, \quad z = \frac{x}{a}$$

$$r\,\mathbf{E}_x[T_a] = 2R\int_{2Rd}^{2Rdz} y^{-2R-1}e^{y}\left(\int_y^{+\infty} t^{2R-1}e^{-t}\exp\left(2R\int_y^t \frac{\beta(\frac{Kv}{2R})}{v}dv\right)dt\right)dy$$

$$r^2\mathbf{VAR}_x[T_a] = 8R^2\int_{2Rd}^{2Rdz} y^{-2R-1}e^{y}\int_y^{+\infty} u^{-2R-1}e^{u}\left(\int_u^{+\infty} t^{2R-1}e^{-t}\exp\left(2R\int_y^t \frac{\beta(\frac{Kv}{2R})}{v}dv\right)dt\right)$$

$$\left(\int_u^{+\infty} t^{2R-1}e^{-t}\exp\left(2R\int_u^t \frac{\beta(\frac{Kv}{2R})}{v}dv\right)dt\right)du\,dy$$

Logistic model

For the logistic model $\alpha(x) = 0$

$$r\ \mathbf{E}_x^{Logistic(R,d)}[T_a] = 2R\int_{2Rd}^{2Rdz} y^{-2R-1} e^{y}\,\Gamma(2R,y)\,dy$$

$$r^2\mathbf{VAR}_x^{Logistic(R,d)}[T_a] = 8R^2\int_{2Rd}^{2Rdz} y^{-2R-1} e^{y}\int_{y}^{+\infty} u^{-2R-1} e^{u}\left(\Gamma(2R,u)\right)^2 du\,dy$$

with $\Gamma(c,x) = \int_{x}^{+\infty} t^{c-1} e^{-t}\,dt$

Logistic model, approximate drift coefficient

$$R^* = R(1-\delta) \qquad d^* = d/(1-\delta)$$
$$R^{**} = R(1+\delta) \qquad d^{**} = d/(1+\delta),$$

$$r\ \mathbf{E}_x[T_a] \geq \frac{1}{1-\delta} 2R^* \int_{2R^*d^*}^{2R^*d^*z} y^{-2R^*-1} e^y \Gamma(2R^*, y) dy = \frac{1}{1-\delta} r\ \mathbf{E}_x^{Logistic(R^*,d^*)}[T_a]$$

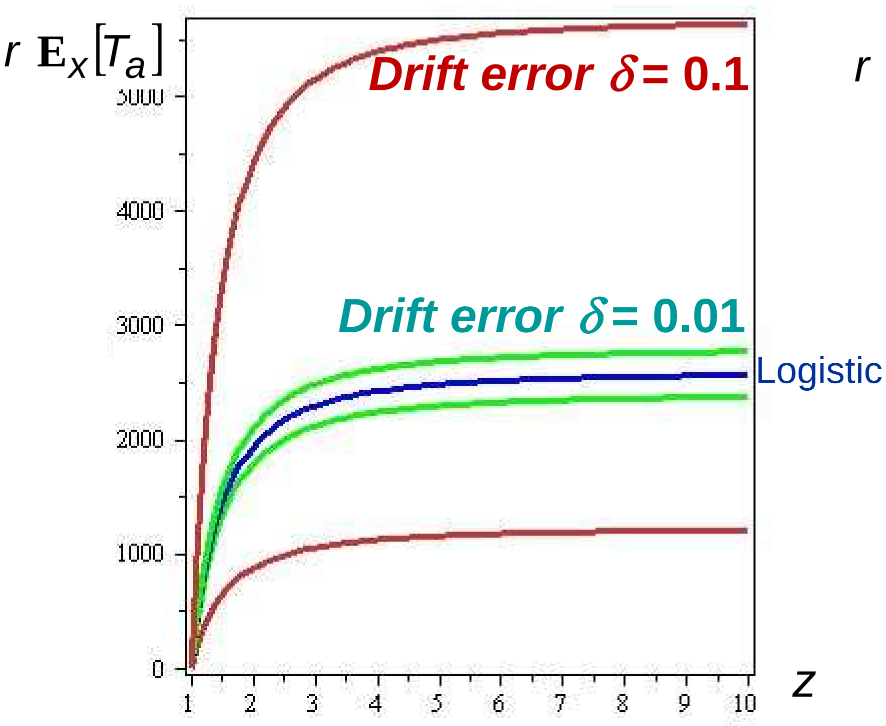
$$r\ \mathbf{E}_x[T_a] \leq \frac{1}{1+\delta} 2R^{**} \int_{2R^{**}d^{**}}^{2R^{**}d^{**}z} y^{-2R^{**}-1} e^y \Gamma(2R^{**}, y) dy = \frac{1}{1+\delta} r\ \mathbf{E}_x^{Logistic(R^{**},d^{**})}[T_a]$$

$$r^2 \mathbf{VAR}_x[T_a] \geq \frac{1}{(1-\delta)^2} 8R^{*2} \int_{2R^*d^*}^{2R^*d^*z} y^{-2R^*-1} e^y \int_y^{+\infty} u^{-2R^*-1} e^u (\Gamma(2R^*, u))^2 du dy$$
$$= \frac{1}{(1-\delta)^2} r^2 \mathbf{VAR}_x^{Logistic(R^*,d^*)}[T_a]$$

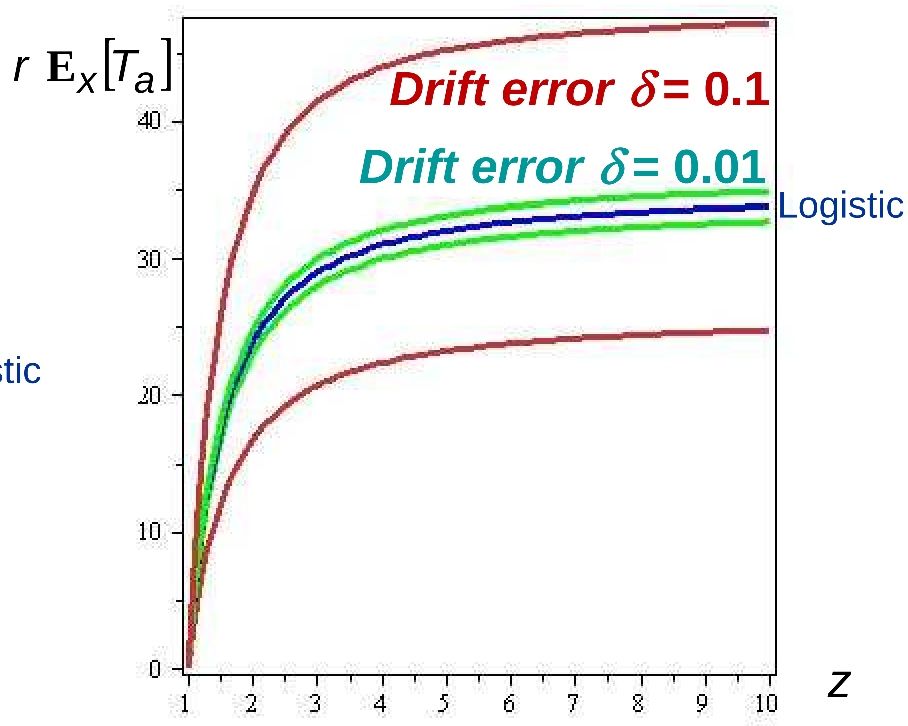
$$r^2 \mathbf{VAR}_x[T_a] \leq \frac{1}{(1+\delta)^2} 8R^{**2} \int_{2R^{**}d^{**}}^{2R^{**}d^{**}z} y^{-2R^{**}-1} e^y \int_y^{+\infty} u^{-2R^{**}-1} e^u (\Gamma(2R^{**}, u))^2 du dy$$
$$= \frac{1}{(1+\delta)^2} r^2 \mathbf{VAR}_x^{Logistic(R^{**},d^{**})}[T_a]$$

Example

Behavior of r times the **mean of the population extinction time** as a function of $z=x/a$



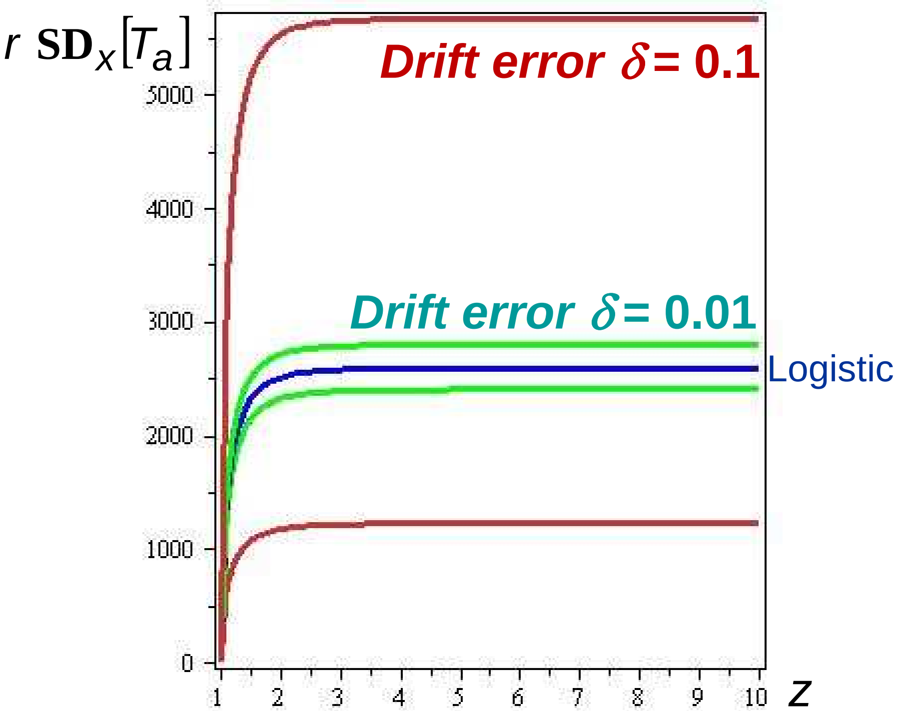
R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)



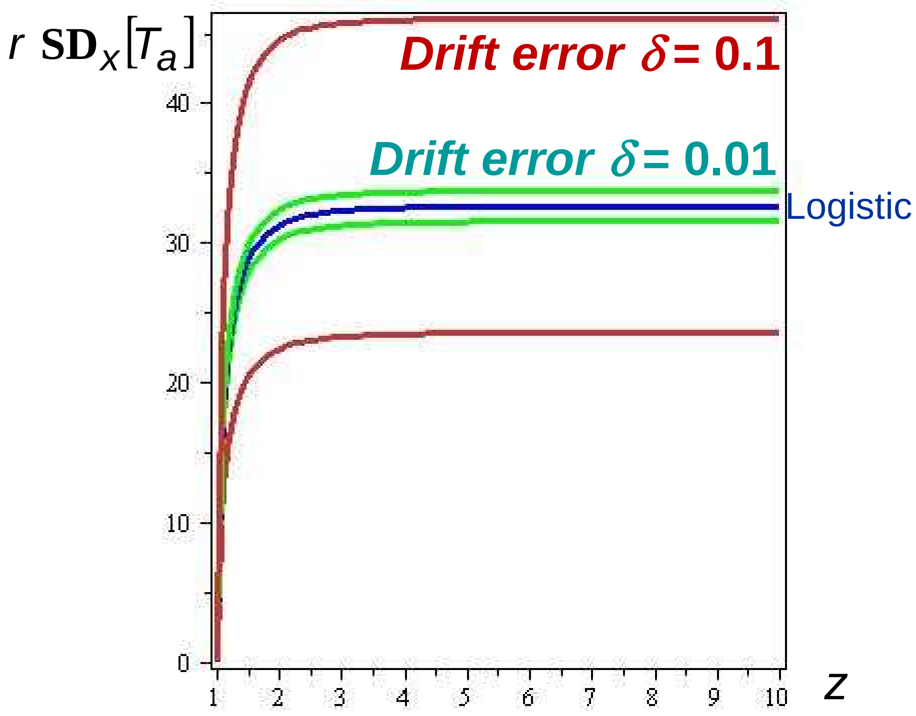
R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Example

Behavior of r times the **standard deviation of the population extinction time** as a function of $z=x/a$



$R=1$ ($r = \sigma^2$)
$d=0.01$ ($a=K/100$)



$R=1$ ($r = \sigma^2$)
$d=0.1$ ($a=K/10$)

Gompertz model, approximate drift coefficient

$$|\beta(X)| = \frac{|\alpha(X)|}{r} \leq \delta, \quad R = \frac{r}{\sigma^2}, \quad d = \frac{a}{K}, \quad z = \frac{x}{a}$$

$$r\ \mathbf{E}_x[T_a] = 2\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} e^{y^2} \int_y^{+\infty} e^{-t^2} \left(\exp\left(2\sqrt{R}\int_y^t \beta\left(K\exp\left(\frac{v}{\sqrt{R}}\right)\right) dv \right) \right) dt dy$$

$$r^2\mathbf{VAR}_x[T_a] = 8\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} e^{y^2} \int_y^{+\infty} e^{u^2} \left(\int_u^{+\infty} e^{-t^2} \left(\exp\left(2\sqrt{R}\int_y^t \beta\left(K\exp\left(\frac{v}{\sqrt{R}}\right)\right) dv \right) \right) dt \right)$$
$$\left(\int_u^{+\infty} e^{-t^2} \left(\exp\left(2\sqrt{R}\int_u^t \beta\left(K\exp\left(\frac{v}{\sqrt{R}}\right)\right) dv \right) \right) dt \right) du dy$$

Gompertz model

For the Gompertz model $\alpha(x) = 0$

$$r\ \mathbf{E}_x^{Gompertz(R,d)}[T_a] = 2\sqrt{\pi}\int_{\sqrt{R}\ln(d)}^{\sqrt{R}\ln(dz)} e^{y^2}\left(1-\Phi(\sqrt{2}y)\right)dy$$

$$r^2\mathbf{VAR}_x^{Gompertz(R,d)}[T_a] = 8\pi\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} e^{y^2}\int_y^{+\infty} e^{u^2}\left(1-\Phi\left(\sqrt{2}u\right)\right)^2 du dy$$

with $\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$ standard Gaussian d.f.

Gompertz model, approximate drift coefficient

$$d^* = d\, e^{\delta}$$

$$d^{**} = d\, e^{-\delta}$$

$$r\ \mathbf{E}_x[T_a] \geq 2\sqrt{\pi}\int_{\sqrt{R}\ln(d^*)}^{\sqrt{R}\ln(d^* z)} e^{y^2}\left(1-\Phi(\sqrt{2}y)\right)dy = r\ \mathbf{E}_x^{Gompertz(R,d^*)}[T_a]$$

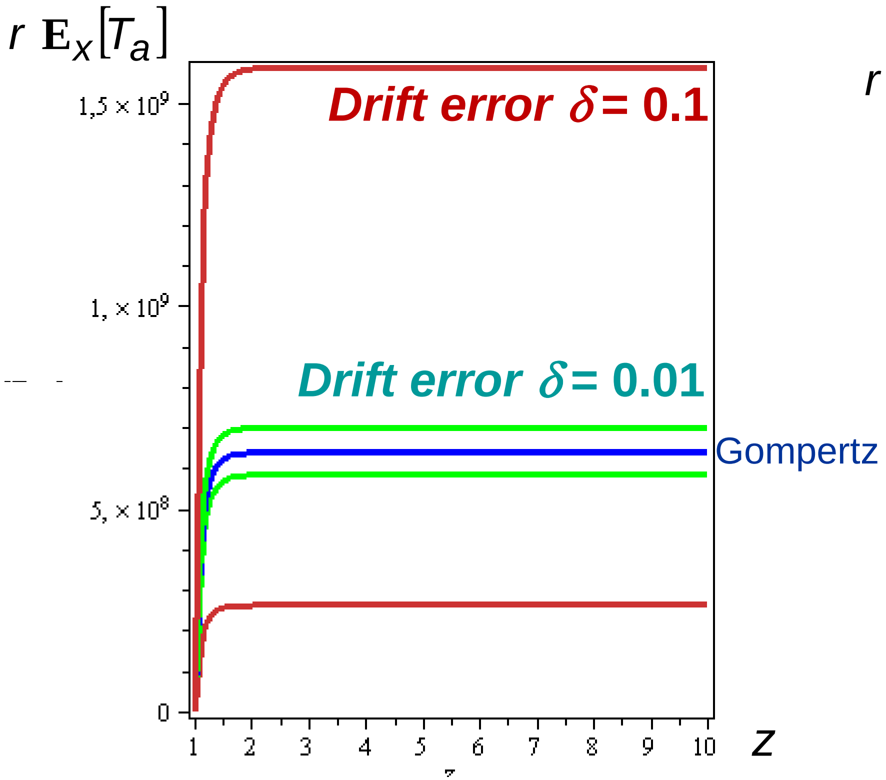
$$r\ \mathbf{E}_x[T_a] \leq 2\sqrt{\pi}\int_{\sqrt{R}\ln(d^{**})}^{\sqrt{R}\ln(d^{**} z)} e^{y^2}\left(1-\Phi(\sqrt{2}y)\right)dy = r\ \mathbf{E}_x^{Gompertz(R,d^{**})}[T_a]$$

$$r^2 Var_x[T_a] \geq 8\pi\int_{\sqrt{R}\ln d^*}^{\sqrt{R}\ln(d^* z)} e^{y^2}\int_y^{+\infty} e^{u^2}\left(1-\Phi\left(\sqrt{2}u\right)\right)^2 du dy = r^2\mathbf{VAR}_x^{Gompertz(R,d^*)}[T_a]$$

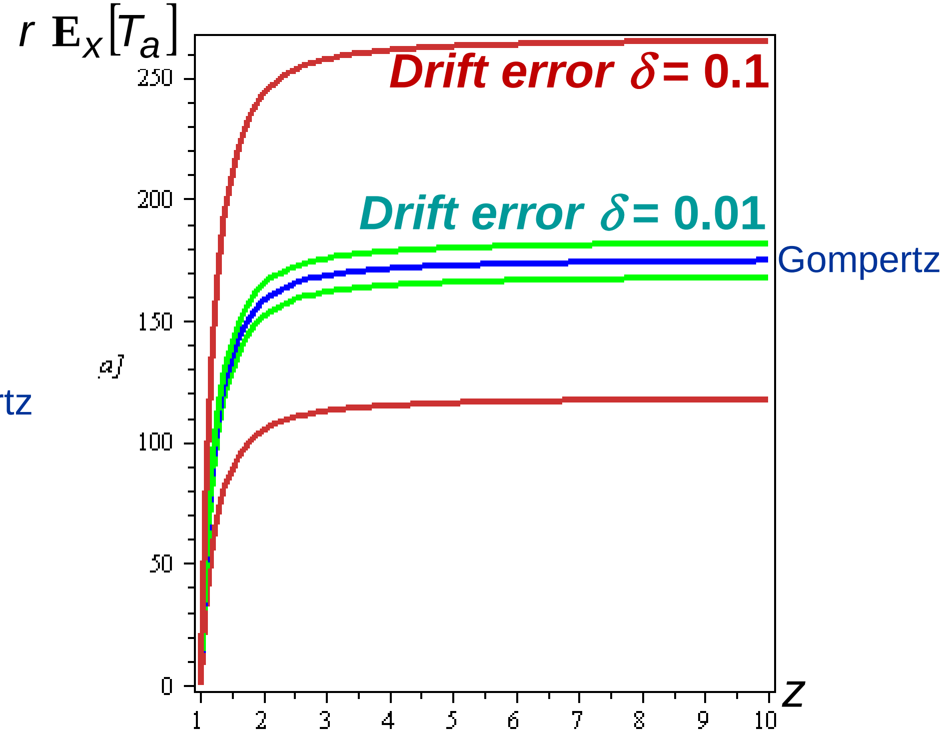
$$r^2\mathbf{VAR}_x[T_a] \leq 8\pi\int_{\sqrt{R}\ln d^{**}}^{\sqrt{R}\ln(d^{**} z)} e^{y^2}\int_y^{+\infty} e^{u^2}\left(1-\Phi\left(\sqrt{2}u\right)\right)^2 du dy = r^2\mathbf{VAR}_x^{Gompertz(R,d^{**})}[T_a]$$

Example

Behavior of *r* times the **mean of the population extinction time** as a function of $z=x/a$



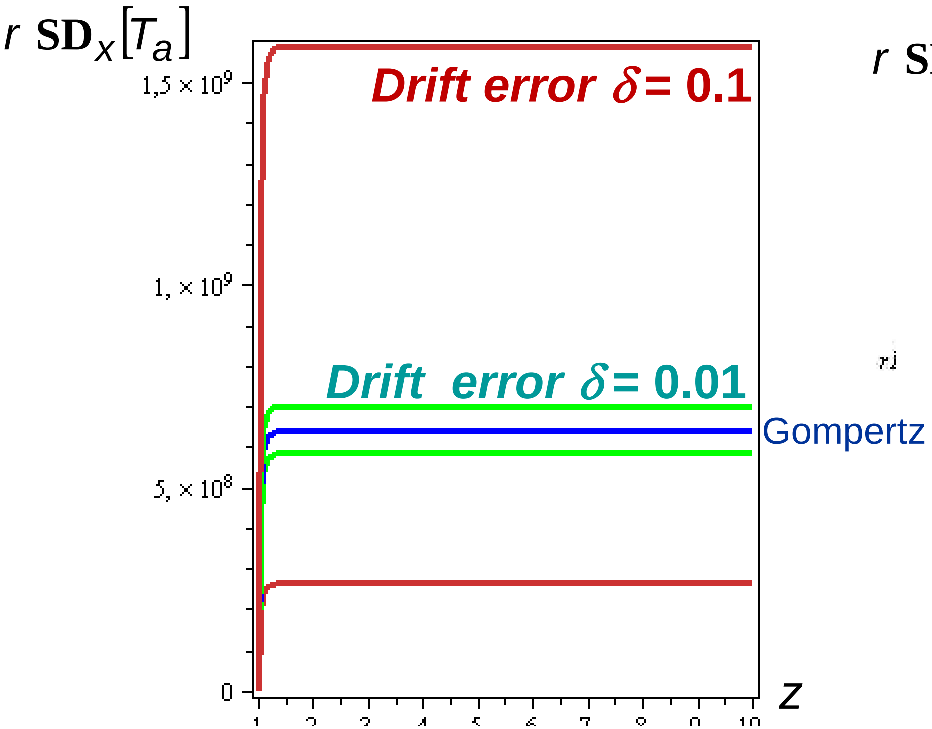
R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)

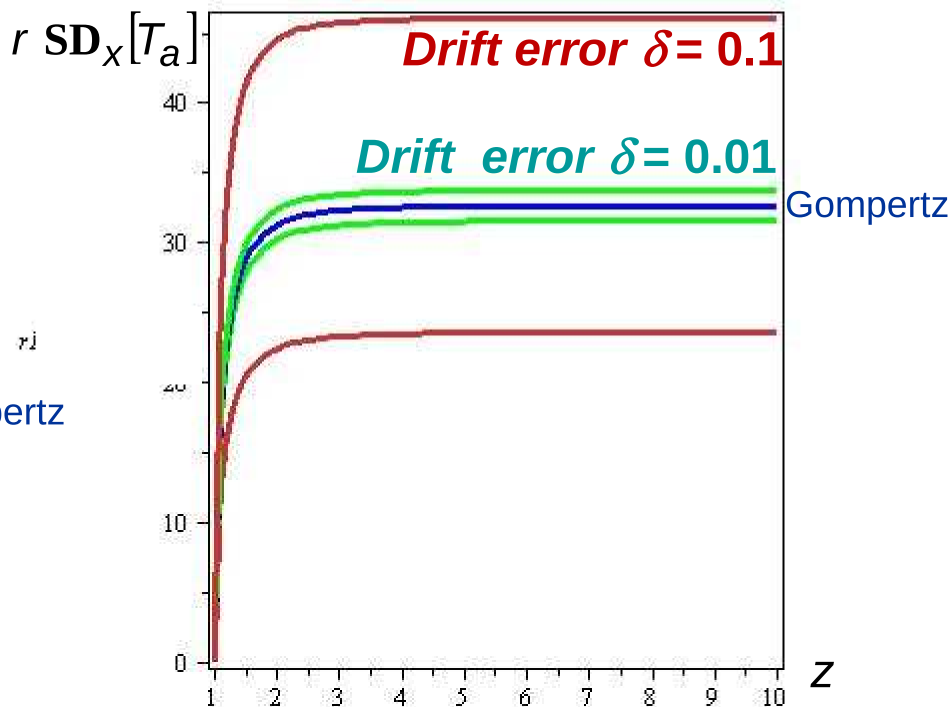


R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Example

Behavior of r times the **standard deviation of the population extinction time** as a function of $z=x/a$





R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)

R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Logistic model, approximate diffusion coefficient

$$|\beta(X)| = \frac{|\alpha(X)|}{\sigma} \le \delta, \quad R = \frac{r}{\sigma^2}, \quad d = \frac{a}{K}, \quad z = \frac{x}{a}$$

$$r\,\mathbf{E}_x[T_a] = 2R\int_{2Rd}^{2Rdz} \frac{1}{\left(1+\beta\left(\frac{Ky}{2r}\right)\right)y} \int_y^{+\infty} \frac{1}{\left(1+\beta\left(\frac{Kt}{2r}\right)\right)t} \exp\left(\int_y^t \frac{\frac{2R}{v}-1}{\left(1+\beta\left(\frac{Kv}{2r}\right)\right)^2} dv\right) dtdy$$

$$r^2\mathbf{VAR}_x[T_a] = 8R^2\int_{2Rd}^{2Rdz} \frac{1}{\left(1+\beta\left(\frac{Ky}{2r}\right)\right)y} \int_y^{+\infty} \frac{1}{\left(1+\beta\left(\frac{Ku}{2r}\right)\right)u} \left(\int_u^{+\infty} \frac{1}{\left(1+\beta\left(\frac{Kt}{2r}\right)\right)t} \exp\left(\int_y^t \frac{\frac{2R}{v}-1}{\left(1+\beta\left(\frac{Kv}{2r}\right)\right)^2} dv\right) dt\right.$$
$$\left.\left(\int_u^{+\infty} \frac{1}{\left(1+\beta\left(\frac{Kt}{2r}\right)\right)t} \exp\left(\int_u^t \frac{\frac{2R}{v}-1}{\left(1+\beta\left(\frac{Kv}{2r}\right)\right)^2} dv\right) dt\right) dudy\right)$$

Logistic model

For the logistic model $\alpha(x) = 0$

$$r\ \mathbf{E}_x^{Logistic(R,d)}[T_a] = 2R\int_{2Rd}^{2Rdz} y^{-2R-1} e^{y}\,\Gamma(2R,y)\,dy$$

$$r^2\mathbf{VAR}_x^{Logistic(R,d)}[T_a] = 8R^2\int_{2Rd}^{2Rdz} y^{-2R-1} e^{y}\int_{y}^{+\infty} u^{-2R-1} e^{u}\left(\Gamma(2R,u)\right)^2 du\,dy$$

with $\Gamma(c,x) = \int_x^{+\infty} t^{c-1} e^{-t}\,dt$

Logistic model, approximate diffusion coefficient

$$R^* = R/(1+\delta)^2 \qquad d^* = d(1+\delta)^2/(1-\delta)^2$$

$$R^{**} = R/(1-\delta)^2 \qquad d^{**} = d(1-\delta)^2/(1+\delta)^2$$

$$r\ \mathbf{E}_x[T_a] \geq 2R^* \int_{2R^*d^*}^{2R^*d^*z} y^{-2R^*-1} e^y \Gamma(2R^*, y)\,dy = r\ \mathbf{E}_x^{Logistic(R^*,d^*)}[T_a]$$

$$r\ \mathbf{E}_x[T_a] \leq 2R^{**} \int_{2R^{**}d^{**}}^{2R^{**}d^{**}z} y^{-2R^{**}-1} e^y \Gamma(2R^{**}, y)\,dy = r\ \mathbf{E}_x^{Logistic(R^{**},d^{**})}[T_a]$$

$$r^2 \mathbf{VAR}_x[T_a] \geq 8R^{*2} \int_{2R^*d^*}^{2R^*d^*z} y^{-2R^*-1} e^y \int_y^{+\infty} u^{-2R^*-1} e^u \left(\Gamma(2R^*, u)\right)^2 du\,dy$$

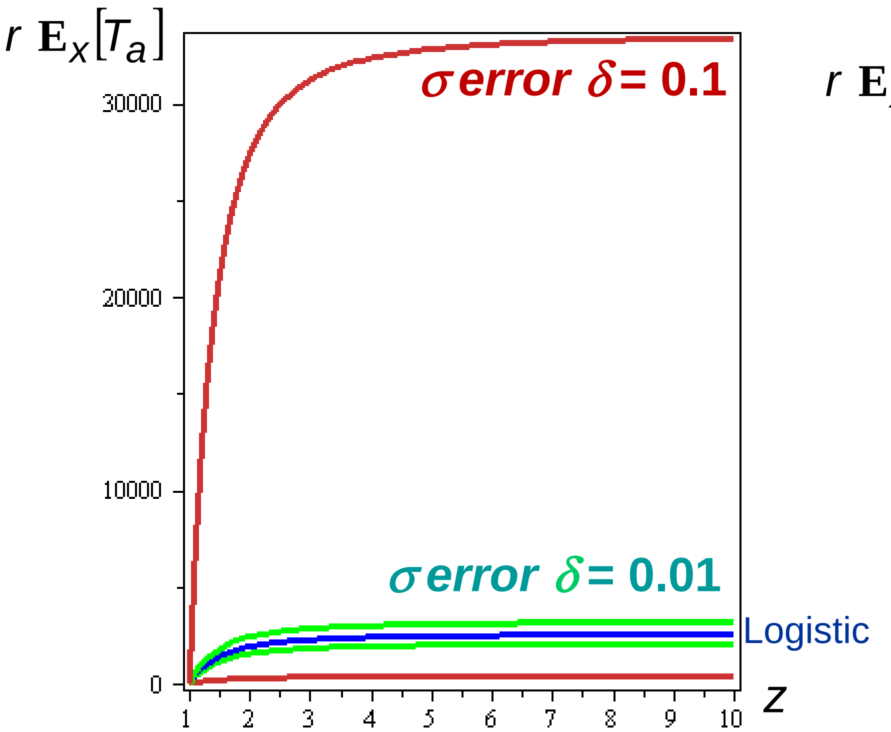
$$= \frac{1}{(1-\delta)^2} r^2 \mathbf{VAR}_x^{Logistic(R^*,d^*)}[T_a]$$

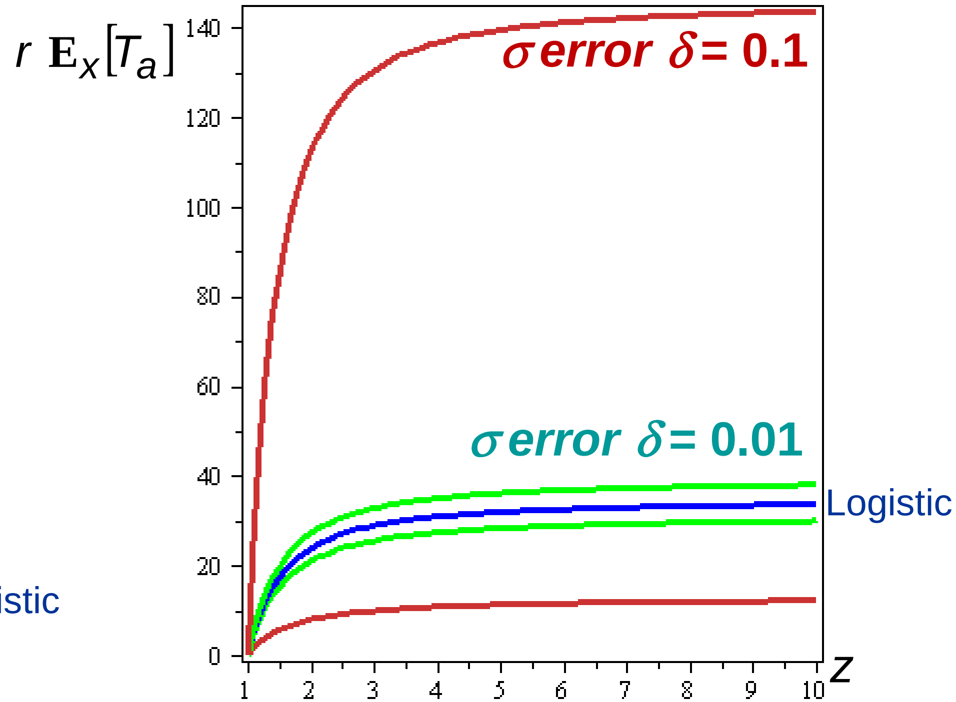
$$r^2 \mathbf{VAR}_x[T_a] \leq 8R^{**2} \int_{2R^{**}d^{**}}^{2R^{**}d^{**}z} y^{-2R^{**}-1} e^y \int_y^{+\infty} u^{-2R^{**}-1} e^u \left(\Gamma(2R^{**}, u)\right)^2 du\,dy$$

$$= \frac{1}{(1+\delta)^2} r^2 \mathbf{VAR}_x^{Logistic(R^{**},d^{**})}[T_a]$$

Example

Behavior of *r* times the **mean of the population extinction time** as a function of *z=x/a*



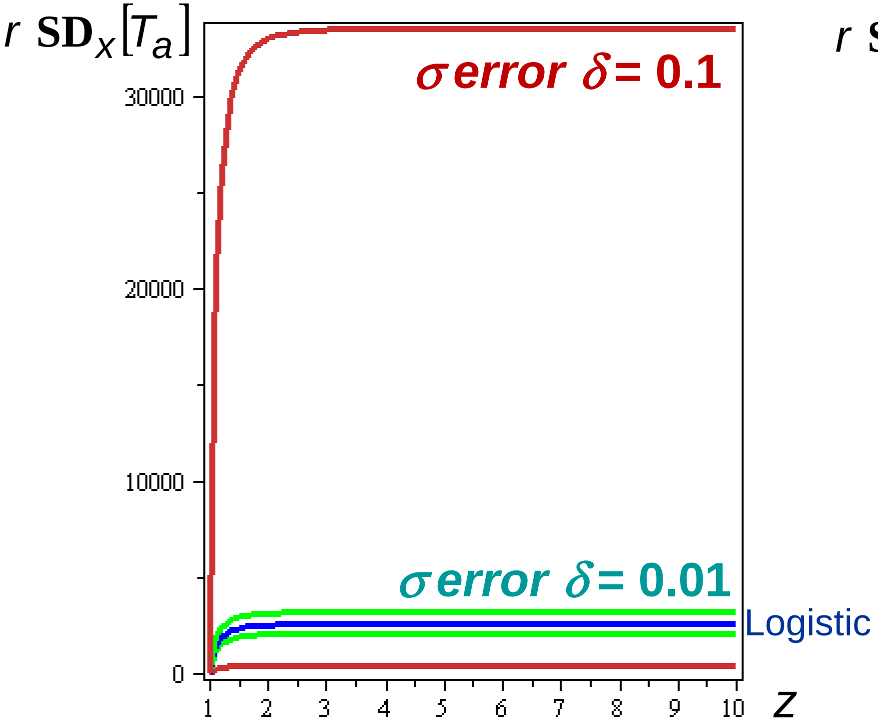


R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)

R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Example

Behavior of r times the **standard deviation of the population extinction time** as a function of $z=x/a$



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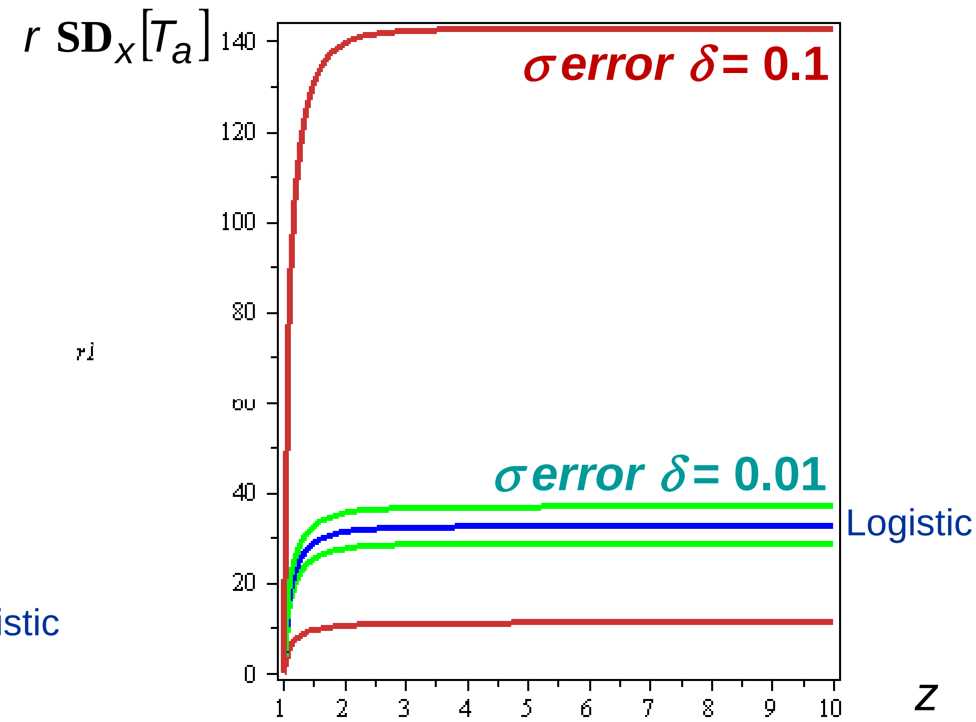


R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Gompertz model, approximate diffusion coefficient

$$|\beta(X)| = \frac{|\alpha(X)|}{\sigma} \le \delta, \quad R = \frac{r}{\sigma^2}, \quad d = \frac{a}{K}, \quad z = \frac{x}{a}$$

$$r\,\mathbf{E}_x[T_a] = 2\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \frac{1}{\left(1+\beta\left(K\exp\left(y/\sqrt{R}\right)\right)\right)} \int_y^{+\infty} \frac{1}{\left(1+\beta\left(K\exp\left(t/\sqrt{R}\right)\right)\right)} \exp\left(-2\int_y^t \frac{v}{\left(1+\beta\left(K\exp\left(v/\sqrt{R}\right)\right)\right)^2} dv\right) dt\,dy$$

$$r^2\mathbf{VAR}_x[T_a] = 8\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \frac{1}{\left(1+\beta\left(K\exp\left(y/\sqrt{R}\right)\right)\right)} \int_y^{+\infty} \frac{1}{\left(1+\beta\left(K\exp\left(u/\sqrt{R}\right)\right)\right)}$$
$$\left(\int_u^{+\infty} \frac{1}{\left(1+\beta\left(K\exp\left(t/\sqrt{R}\right)\right)\right)} \exp\left(-2\int_y^t \frac{v}{\left(1+\beta\left(K\exp\left(v/\sqrt{R}\right)\right)\right)^2} dv\right) dt\right)$$
$$\left(\int_u^{+\infty} \frac{1}{\left(1+\beta\left(K\exp\left(t/\sqrt{R}\right)\right)\right)} \exp\left(-2\int_u^t \frac{v}{\left(1+\beta\left(K\exp\left(v/\sqrt{R}\right)\right)\right)^2} dv\right) dt\right) du\,dy$$

Gompertz model

For the Gompertz model $\alpha(x) = 0$

$$r\ \mathbf{E}_x^{Gompertz(R,d)}[T_a] = 2\sqrt{\pi}\int_{\sqrt{R}\ln(d)}^{\sqrt{R}\ln(dz)} e^{y^2}\left(1-\Phi(\sqrt{2}y)\right)dy$$

$$r^2\mathbf{VAR}_x^{Gompertz(R,d)}[T_a] = 8\pi\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} e^{y^2}\int_y^{+\infty} e^{u^2}\left(1-\Phi\left(\sqrt{2}u\right)\right)^2 du dy$$

with $\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$ standard Gaussian d.f.

Gompertz model, approximate diffusion coefficient

$$r\,\mathbf{E}_x[T_a] \geq \frac{2}{(1+\delta)^2}\sqrt{\pi}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1+\delta)^2}\right)\left(\int_y^0 \exp\left(-\frac{t^2}{(1+\delta)^2}\right)dt + \int_0^{+\infty}\exp\left(-\frac{t^2}{(1-\delta)^2}\right)dt\right)dy$$

$$r\,\mathbf{E}_x[T_a] \leq \frac{2}{(1-\delta)^2}\sqrt{\pi}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1-\delta)^2}\right)\left(\int_y^0 \exp\left(-\frac{t^2}{(1-\delta)^2}\right)dt + \int_0^{+\infty}\exp\left(-\frac{t^2}{(1+\delta)^2}\right)dt\right)dy$$

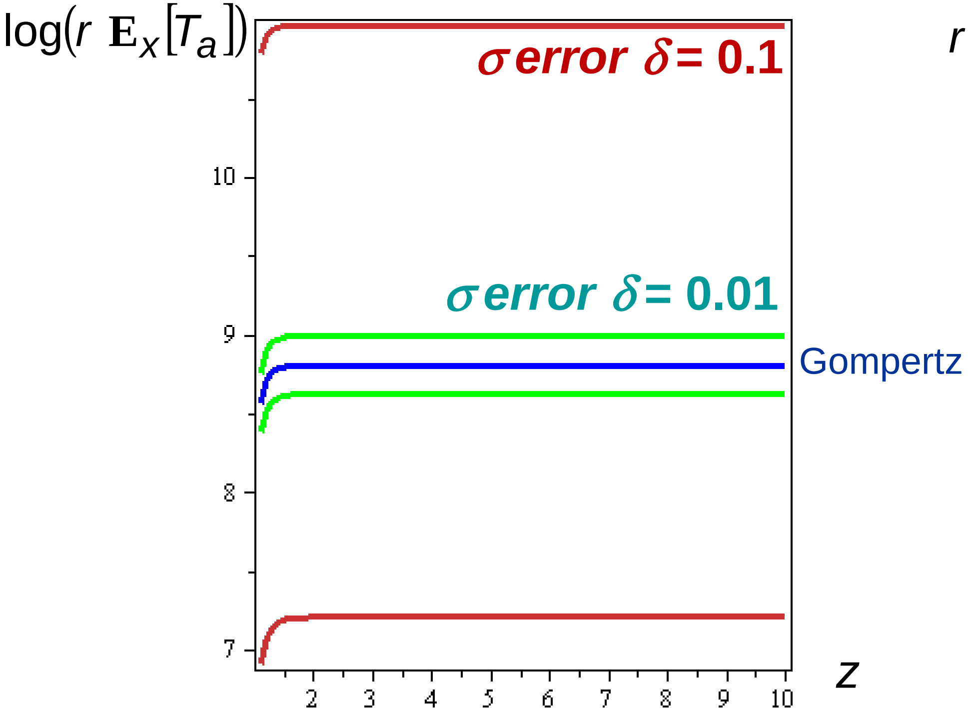
$$r^2\mathbf{VAR}_x[T_a] \geq \frac{8}{(1+\delta)^4}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1+\delta)^2}\right)\int_y^0 \exp\left(\frac{u^2}{(1+\delta)^2}\right)\left(\int_u^0 \exp\left(-\frac{t^2}{(1+\delta)^2}\right)dt + \int_0^{+\infty}\exp\left(-\frac{t^2}{(1-\delta)^2}\right)dt\right)^2 du\,dy$$
$$+\frac{8}{(1-\delta)^4}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1+\delta)^2}\right)\int_0^{+\infty} \exp\left(\frac{u^2}{(1-\delta)^2}\right)\left(\int_u^{+\infty} \exp\left(-\frac{t^2}{(1-\delta)^2}\right)dt\right)^2 du\,dy$$

$$r^2\mathbf{VAR}_x[T_a] \leq \frac{8}{(1-\delta)^4}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1-\delta)^2}\right)\int_y^0 \exp\left(\frac{u^2}{(1-\delta)^2}\right)\left(\int_u^0 \exp\left(-\frac{t^2}{(1-\delta)^2}\right)dt + \int_0^{+\infty}\exp\left(-\frac{t^2}{(1+\delta)^2}\right)dt\right)^2 du\,dy$$
$$+\frac{8}{(1-\delta)^4}\int_{\sqrt{R}\ln d}^{\sqrt{R}\ln(dz)} \exp\left(\frac{y^2}{(1-\delta)^2}\right)\int_0^{+\infty} \exp\left(\frac{u^2}{(1+\delta)^2}\right)\left(\int_u^{+\infty} \exp\left(-\frac{t^2}{(1+\delta)^2}\right)dt\right)^2 du\,dy$$

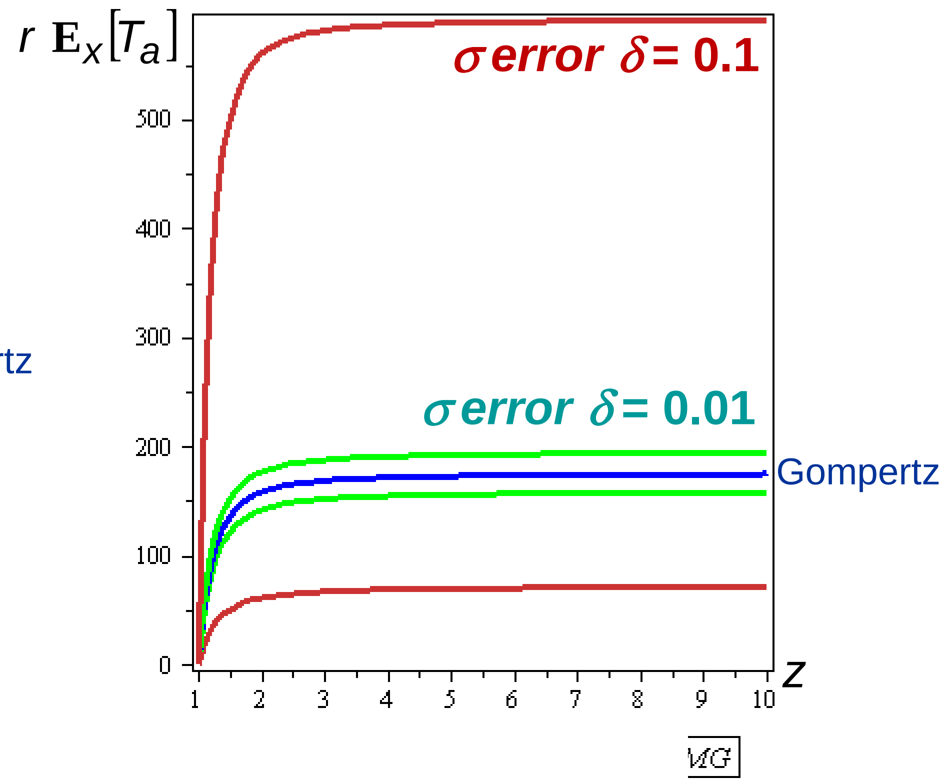
There are “nicer” but not so good bounds

Example

Behavior of r times the **mean of the population extinction time** as a function of $z=x/a$



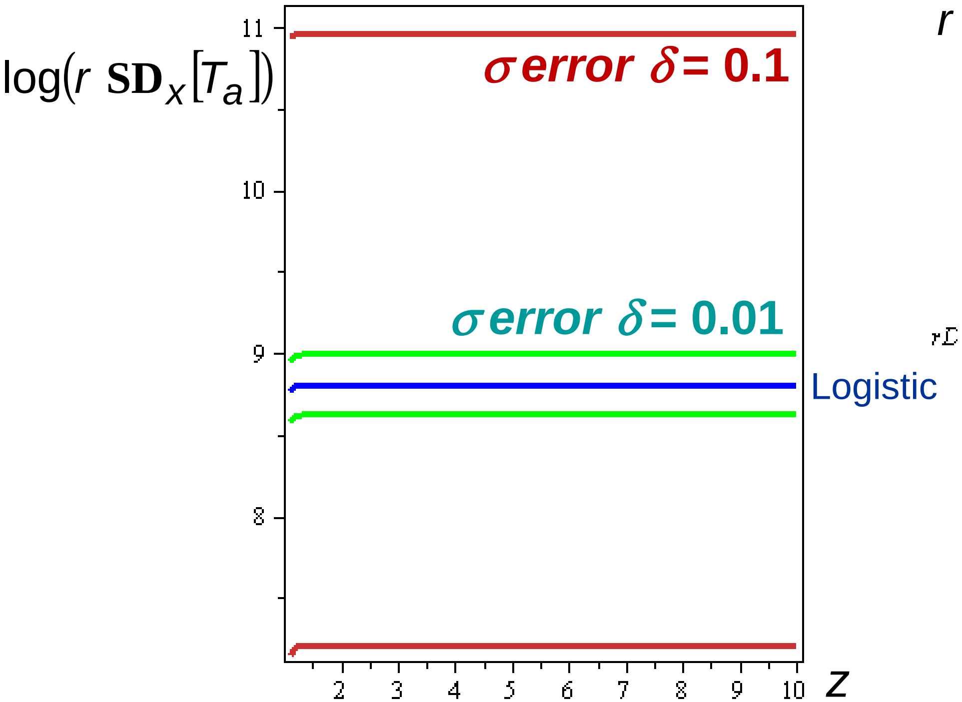
R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)



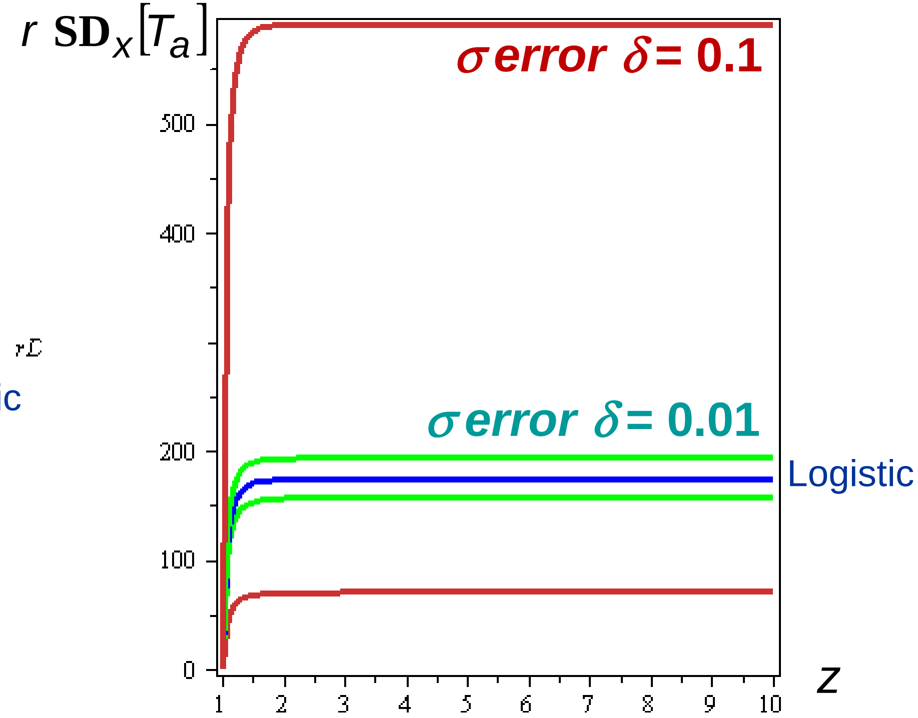
R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Example

Behavior of r times the **standard deviation of the population extinction time** as a function of $z=x/a$



R=1 ($r = \sigma^2$)
d=0.01 (a=K/100)



R=1 ($r = \sigma^2$)
d=0.1 (a=K/10)

Conclusions

- The qualitative behavior of the "true" models coincides with the bahavior of the approximate standard models (logistic or Gompertz). "Mathematical" extinction has zero probability of occurring and there is a staionary density. "Realistic" extinction occurs with probability one.

- If the true average growth rate or the true environmental noise intensity are very close to the standard model (logistic or Gompertz), the mean and standard deviation of the population extinction time are close to the ones of the standard model. One can then use a standard model, which is much simpler to deal with, as a convenient approximation. One can even have bounds for the error committed.

- Otherwise, the use of the true model may lead to values quite different from the true ones.

- The standard deviation of the population extinction time is of the same order of magnitude as the mean extinction time. So, use of means alone is not very informative.

References

- Braumann, C. A. (1999). Aplications of stochastic differential equations to population growth. Proceeding of the Ninth International Colloquium on Differential Equations, (D. Bainov, ed), VSP, Utrecht, p. 47--52.
- Braumann, C. A. (2005). *Introdução às Equações Diferenciais Estocásticas*. Edições SPE, Lisboa.
- Braumann, C. A. (2008). Growth and extinction of populations in randomly varying environments. *Computers and Mathematics with Appl.* 56: 631-644.
- Braumann, C. A., Filipe, P. A., Carlos C. e Roquete, C. J. (2009). Growth of individuals in randomly fluctuating environments. Proceedings of the International Conference in Computational and Mathematical Methods in Science e Engineering}, Vigo-Aguiar, J., Alonso, P., Oharu, S., Venturino, E. and Wade, B. (Eds.), Gijon, p. 201--212.
- Carlos, C. e Braumann, C. A. (2005). Tempos de extinção para populações em ambiente aleatório. Estat\'{i}stica Jubilar, Braumann, C. A., Infante, P., Oliveira, M., Alpízar-Jara, R. e Rosado, F. (Eds.), Ediçóes SPE, p. 133—142.
- Carlos, C. and Braumann, C. A. (2006). Tempos de extinção para populações em ambiente aleatório e cálculos de Itô e Stratonovich. In *Ciência Estatística* (L. Canto e Castro, E. G. Martins, C. Rocha, M. F.Oliveira, M. M. Leal, and F. Rosado, eds.). Edições SPE, Lisboa, p. 229-238.
- Ghiman, I. I. and Skorohod, A. V. (1991). Stochastic Differential Equations. Springer, Berlin.
- Karlin, S. and Taylor, H. M. (1981). A Second Course in Stochastic Processes. Academic Press, New York.

Thank you.