Analytical solution of the *n*-site approximation of the majority-vote model for the spread of rumors in a chain

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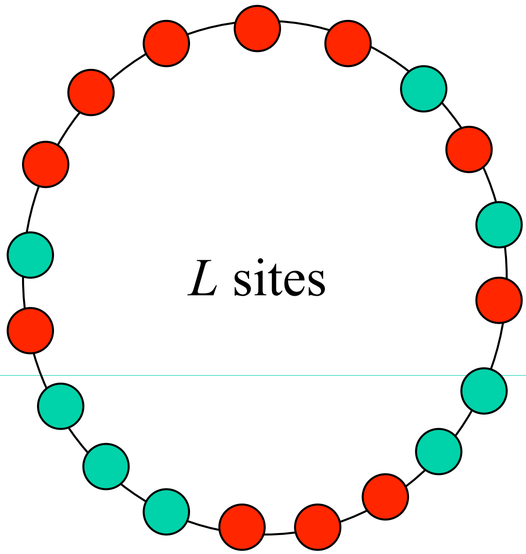
Model



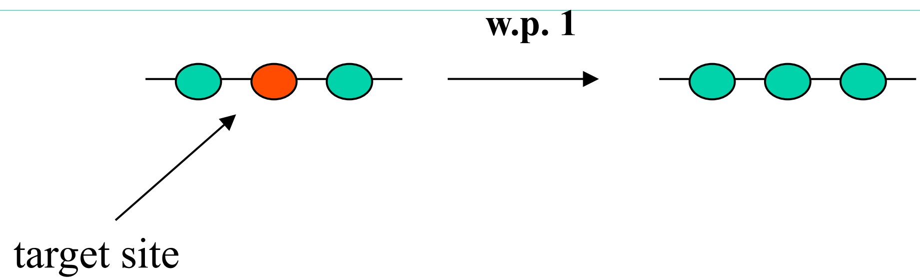
•Individuals represented by their opinions

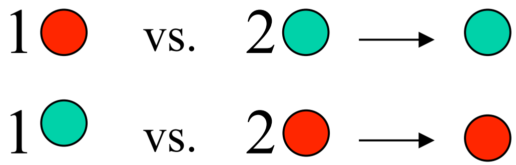
pro $\sigma_i = 1$

con $\sigma_i = 0$

Ising model

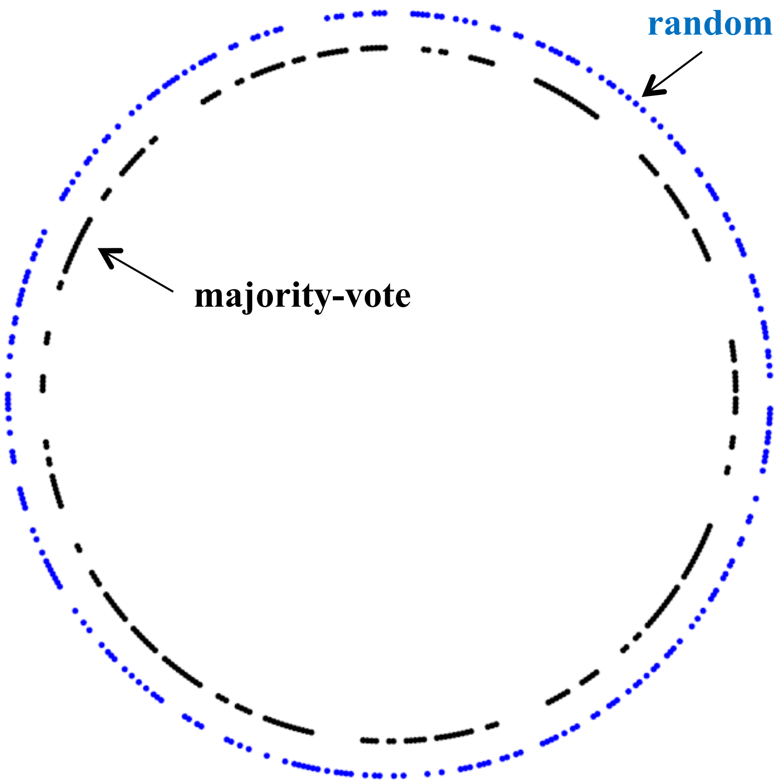
•Interactions between neighbors





majority-vote rule

The outcome



dynamics always freezes in a heterogeneous absorbing configuration!!

(majority-vote = Ising + Glauber dynamics)

????

ergodicity breaking makes the model “nontrivial”

Some formalism...

ring configuration

$$\sigma = (\sigma_1, \ldots, \ldots, \sigma_L)$$

master equation

$$\frac{d}{dt} P(\sigma, t) = \sum_i \left[W_i(\tilde{\sigma}^i) P(\tilde{\sigma}^i, t) - W_i(\sigma) P(\sigma, t) \right]$$

$$\tilde{\sigma}^i \;=\; (\sigma_1, \ldots, 1 - \sigma_i, \ldots, \sigma_L)$$

transition rates

$$W_i(\sigma) = \left| \Theta\left[\sigma_{i-1} + \sigma_i + \sigma_{i+1} - 1\right] - \sigma_i \right| \quad \longrightarrow \quad 0 \text{ or } 1$$

useful expectations

$$\frac{d}{dt}\langle\sigma_i\rangle = \langle(1-2\sigma_i)\,W_i(\sigma)\rangle$$

$$\langle\sigma_i\rangle = \rho$$

$$\frac{d}{dt}\langle\sigma_i\sigma_j\rangle = 2\,\langle\sigma_j\,(1-2\sigma_i)\,W_i(\sigma)\rangle$$

$$\langle\sigma_i\sigma_j\rangle = \phi$$

$$\frac{d}{dt}\langle\sigma_i\sigma_j\sigma_k\rangle = 3\,\langle\sigma_j\sigma_k\,(1-2\sigma_i)\,W_i(\sigma)\rangle$$

$$\langle\sigma_i\sigma_j\sigma_k\rangle = \psi$$

...

$$\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle = \omega$$

The pair approximation

$$P(\sigma) = \frac{p_2(\sigma_1,\sigma_2)\,p_2(\sigma_2,\sigma_3)\ldots p_2(\sigma_{L-1},\sigma_L)\,p_2(\sigma_L,\sigma_1)}{p_1(\sigma_1)\,p_1(\sigma_2)\ldots p_1(\sigma_{L-1})\,p_1(\sigma_L)}$$

with $p_1(\sigma_i) = \sum_{\sigma_j} p_2(\sigma_i,\sigma_j)$

$p_2(\sigma_i,\sigma_{i+1}) \longrightarrow$ 4 unknowns $\left\{ \begin{array}{l} p_2(1,1) \\ p_2(1,0) \\ p_2(0,1) \\ p_2(0,0) \end{array} \right.$

parity symmetry
normalization $\longrightarrow \left\{ \begin{array}{l} p_2(1,1) = \langle\sigma_i\sigma_{i+1}\rangle = \phi \\ p_2(1,0) = p_1(1) - p_2(1,1) = \rho - \phi \end{array} \right.$

ϕ

ρ

Self-consistent equations

$$\dot{\rho} = \frac{(\rho - \phi)^2 (2\rho - 1)}{2\rho (1 - \rho)}$$

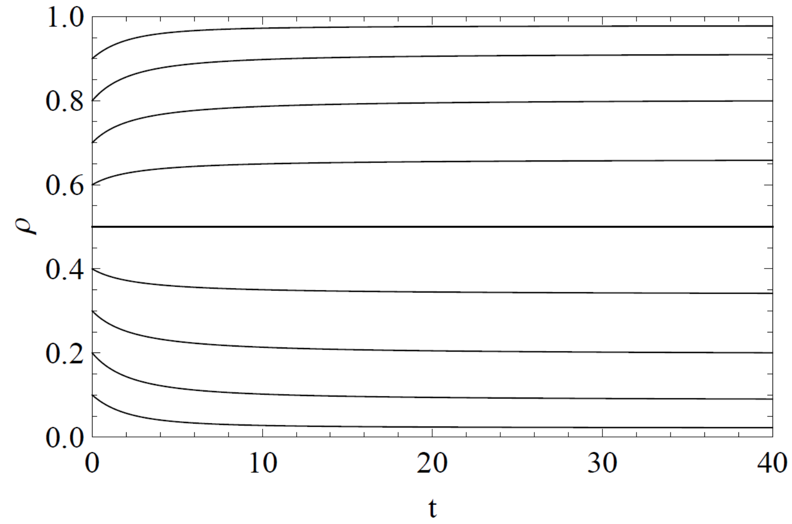
$$\dot{\phi} = \frac{(\rho - \phi)^2}{1 - \rho}$$

steady-state:

$$\rho = \phi$$

but what is ρ?

numerical solution of the ODE:



a continuous of fixed points!

$$\bar{\rho}(\rho_0)$$

ergodicity breaking within a mean-field framework!

Solution of $\bar{\rho}\left(\rho_0\right)$

$$x \equiv p_2\left(1,0\right) = \rho - \phi$$ vanishes at the steady state

$$\left.\begin{aligned} \dot{\rho} &= \frac{x^2\left(2\rho-1\right)}{2\rho\left(1-\rho\right)} \\ \dot{x} &= -\frac{x^2}{2\rho\left(1-\rho\right)} \end{aligned}\right\} \quad \int_{x_0}^{x(t)} dx' = -\int_{\rho_0}^{\rho(t)} \frac{d\rho'}{2\rho'-1}$$

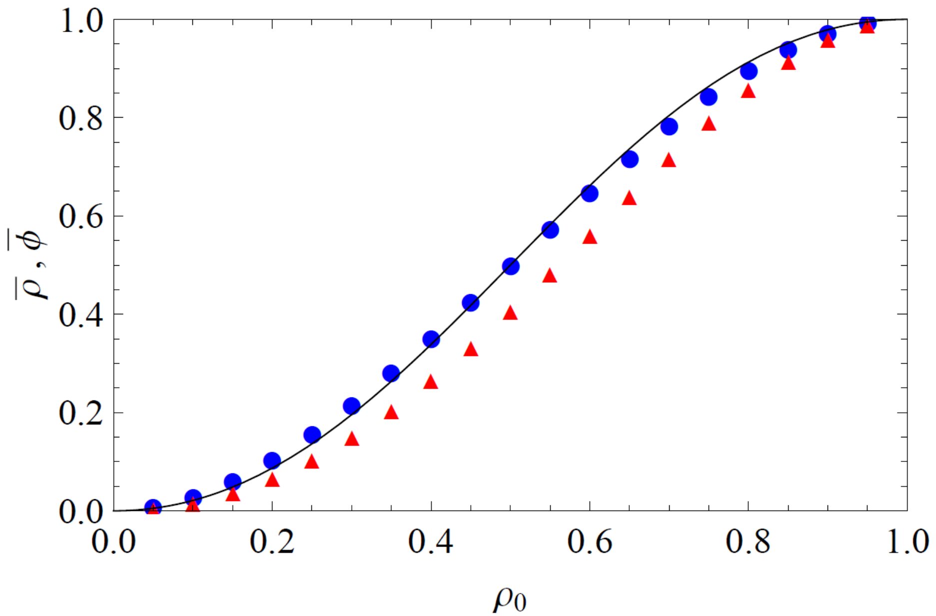
$$\rho\left(t=0\right) = \rho_0 \quad \phi\left(t=0\right) = \rho_0^2 \quad x_0 = \rho_0\left(1-\rho_0\right)$$

$$\bar{\rho}\left(\rho_0\right) \equiv \rho\left(t\to\infty\right) \quad x\left(t\to\infty\right) = 0$$

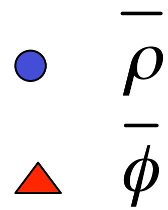
$$\boxed{\bar{\rho}\left(\rho_0\right) = \frac{1}{2}\left[1 + \left(2\rho_0 - 1\right)e^{2\rho_0\left(1-\rho_0\right)}\right]}$$

But $\bar{\phi} = \bar{\rho}$ $\longrightarrow$ $p_2\,(1,0) = 0$ **is nonphysical!**

(unless $\bar{\rho}$ = 0 or 1)



Monte Carlo results



The 3-site approximation

$$P(\sigma) = \frac{p_3(\sigma_1,\sigma_2,\sigma_3)\, p_3(\sigma_2,\sigma_3,\sigma_4)\ldots p_3(\sigma_L,\sigma_1,\sigma_2)}{p_2(\sigma_1,\sigma_2)\, p_2(\sigma_2,\sigma_3)\ldots p_2(\sigma_L,\sigma_1)}$$

independent variables

$$x_0 = p_3(0,0,0), \qquad x_1 = p_3(1,0,0), \qquad x_2 = p_3(1,1,0)$$

$$x_{1C} = p_3(0,1,0), \qquad x_{2C} = p_3(1,0,1), \qquad x_3 = p_3(1,1,1)$$

use of symmetries $x_{1C} + x_2 = x_{2C} + x_1$ and lots of algebra

⇩

$$\rho(t) = \rho_0 - \rho_0(1-\rho_0)(1-2\rho_0)\left(1 - e^{-\frac{1}{3}t}\right)$$

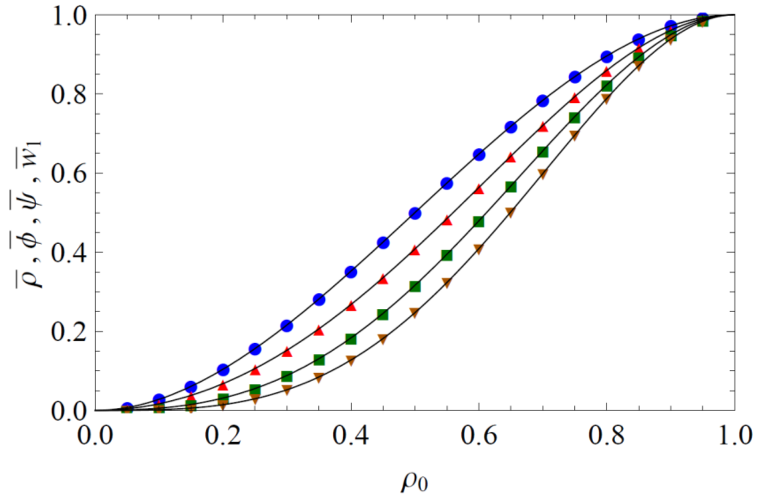
Steady-state solution

$p_3(0,1,0) = p_3(1,0,1)$ physical!!

$$\bar{\rho}(\rho_0) = \rho_0^2(3-2\rho_0)$$

$$\bar{\phi}(\rho_0) = \rho_0^2(3-2\rho_0) - \bar{y}(\rho_0)$$

$$\bar{\psi}(\rho_0) = \rho_0^2(3-2\rho_0) - 2\bar{y}(\rho_0)$$



where

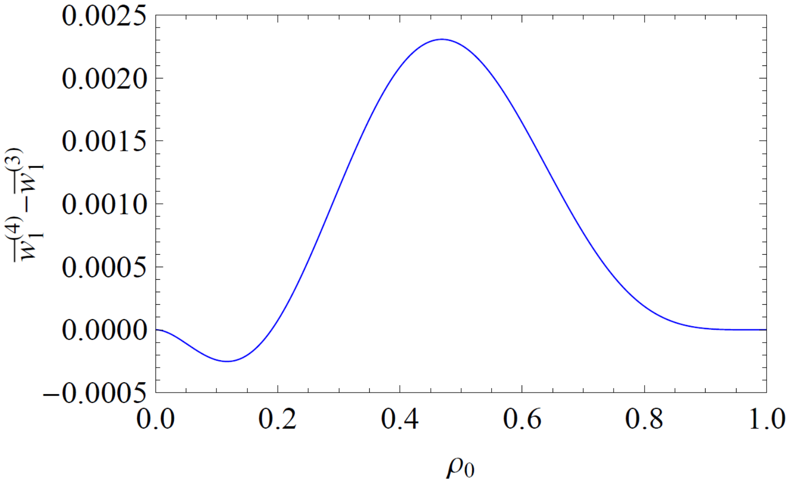
$$\bar{y}(\rho_0) = \rho_0(1-\rho_0)\cosh\left[2\sqrt{\rho_0(1-\rho_0)}\right] - \frac{1}{2}\sqrt{\rho_0(1-\rho_0)}\sinh\left[2\sqrt{\rho_0(1-\rho_0)}\right]$$

The 4-site approximation

$$P(\sigma) = \frac{p_4(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\, p_4(\sigma_2,\sigma_3,\sigma_4,\sigma_5)\ldots p_4(\sigma_L,\sigma_1,\sigma_2,\sigma_3)}{p_3(\sigma_1,\sigma_2,\sigma_3)\, p_3(\sigma_2,\sigma_3,\sigma_4)\ldots p_3(\sigma_L,\sigma_1,\sigma_2)}$$

same results as the 3-site approximation for $\langle\sigma_i\rangle=\rho \quad \langle\sigma_i\sigma_j\rangle=\phi \quad \langle\sigma_i\sigma_j\sigma_k\rangle=\psi$

but NOT for $\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle=\omega$



Conclusion

- Probably exact analytical expressions for

$$\langle \sigma_i \rangle = \rho \qquad \langle \sigma_i \sigma_j \rangle = \phi \qquad \langle \sigma_i \sigma_j \sigma_k \rangle = \psi$$

THANKS