Analysis of Arnold tongues in non-autonomous epidemiological models

Bob W. Kooi[a]

Maíra Aguiar[b], Nico Stollenwerk[b]

[a]Dept. Theoretical Biology,
Vrije Universiteit, de Boelelaan 1085,
1081 HV Amsterdam, The Netherlands

[b]Centro de Matemática e Aplicações Fundamentais
Universidade de Lisboa, Portugal

kooi@bio.vu.nl

http://www.bio.vu.nl/thb/

- Talk Nico Stollenwerk:
  Dynamic noise, chaos and parameter estimation in population biology
  Parameter estimation framework

- Talk Maíra Aguiar:
  How much complexity is needed to describe dengue haemorrhagic fever incidence data
  Model complexity evaluation related to incidence data

- Here: Model analysis with Bifurcation analysis

Outline

- multi-strain epidemiological model
  - local bifurcations, global bifurcations
  - symmetry, Hopf- bifurcation and Torus bifurcation, chaos
- seasonally forced multi-strain epidemiological model
  - Arnold tongues

Bifurcation analysis: Nonlinear Dynamical System Theory

- Continuous-time systems (sets of odes)

- Discrete-time systems (maps)

- Periodically force systems (odes and maps)

Long-term dynamics

- Limit sets: equilibria, limit cycles and chaotic attractors

- Stability of limit sets

Dependency on parameters

- critical parameter values where dynamics changes qualitatively
- continuation of limit sets
- Bifurcations: linearisation, eigenvalues of Jacobian matrix or Lyapunov exponents from time series

Dengue fever model

with antibody-dependent enhancement (ADE)
and temporary cross immunity

$$
\begin{aligned}
\dot{S} &= -\frac{\beta}{N}S(I_1+\phi I_{21}) - \frac{\beta}{N}S(I_2+\phi I_{12}) + \mu(N-S)\\
\dot{I}_1 &= \frac{\beta}{N}S(I_1+\phi I_{21}) - (\gamma+\mu)I_1\\
\dot{I}_2 &= \frac{\beta}{N}S(I_2+\phi I_{12}) - (\gamma+\mu)I_2\\
\dot{R}_1 &= \gamma I_1 - (\alpha+\mu)R_1\\
\dot{R}_2 &= \gamma I_2 - (\alpha+\mu)R_2\\
\dot{S}_1 &= -\frac{\beta}{N}S_1(I_2+\phi I_{12}) + \alpha R_1 - \mu S_1\\
\dot{S}_2 &= -\frac{\beta}{N}S_2(I_1+\phi I_{21}) + \alpha R_2 - \mu S_2\\
\dot{I}_{12} &= \frac{\beta}{N}S_1(I_2+\phi I_{12}) - (\gamma+\mu)I_{12}\\
\dot{I}_{21} &= \frac{\beta}{N}S_2(I_1+\phi I_{21}) - (\gamma+\mu)I_{21}\\
\dot{R} &= \gamma(I_{12}+I_{21}) - \mu R
\end{aligned}
$$

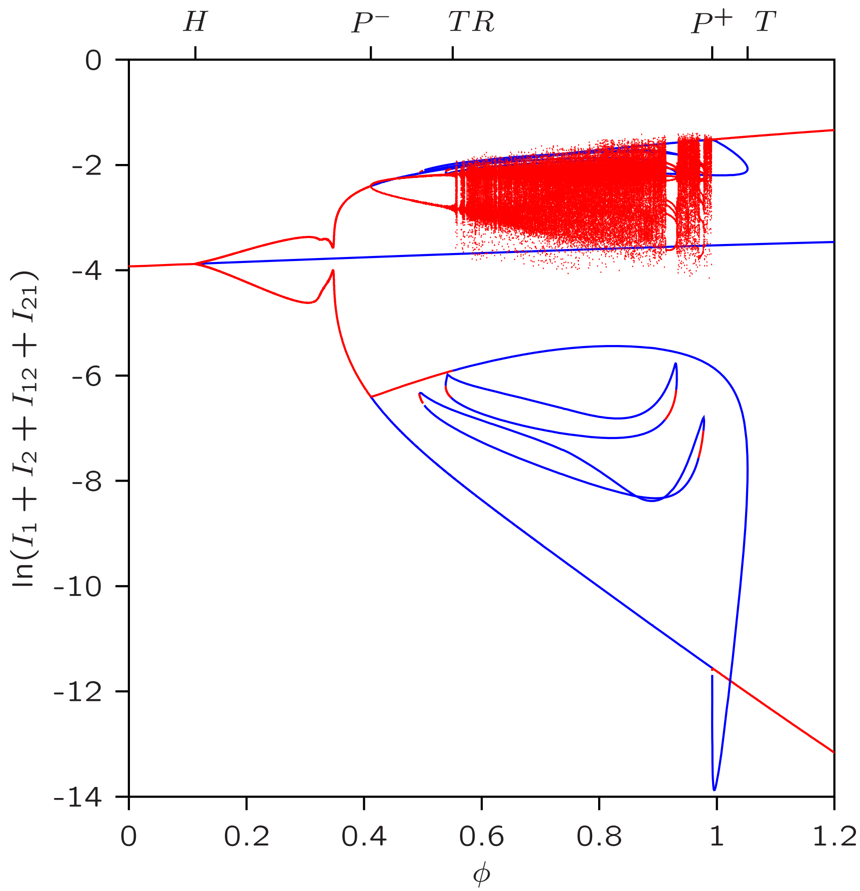
Var.	Description
S	Susceptibles to both strains
I_i	Infected with strain i
R_i	Recovered from infection with strain i
S_i	Immune against first infection strain i but susceptible to j
I_{ij}	S_i Reinfected with strain j
	either by meeting I_2 or by meeting I_{12}
R	Immune to both strains

Two different strains:
$i = 1, j = 2$ and $i = 2, j = 1$

$R = N - (S + I_1 + I_2 + R_1 + R_2 + S_1 + S_2 + I_{12} + I_{21})$ where N is population size

Par.	Description	Values
N	population size	100
μ	new born susceptible rate	$1/65$
γ	recovery rate	52
β_0	infection rate	2γ
α	temporary cross-immunity rate	2, free
ρ	external infected portion	0, free
ϕ	ratio of contribution to force of infection	free
η	seasonal force	$0.1, 0.2, 0.35$, free
T_0	period of system	
T	period of forcing	1 year

One-parameter bifurcation diagram: total infected $I_1 + I_2 + I_{12} + I_{21}$



Stable Unstable

Bifurcations

Symbol	Description bifurcation
Equilibrium	
H	Hopf
Equilibrium, limit cycle	
T	Tangent (saddle node)
P	Pitchfork
Limit cycle	
TR	Torus (Neimark-Sacker)

Symmetries

Symmetries due to the multi-strain structure of the model

Symmetry transformation matrix $\mathbf{S}$

$$\mathbf{S} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} S \\ I_1 \\ I_2 \\ R_1 \\ R_2 \\ S_1 \\ S_2 \\ I_{12} \\ I_{21} \\ R \end{matrix}$$

We have the following symmetry:

$$\underline{x}^* = \begin{pmatrix} S^* \\ I_1^* \\ I_2^* \\ R_1^* \\ R_2^* \\ S_1^* \\ S_2^* \\ I_{12}^* \\ I_{21}^* \\ R^* \end{pmatrix} \quad \Rightarrow \quad \mathbf{S}\underline{x}^* = \begin{pmatrix} S^* \\ I_2^* \\ I_1^* \\ R_2^* \\ R_1^* \\ S_2^* \\ S_1^* \\ I_{21}^* \\ I_{12}^* \\ R^* \end{pmatrix}$$

For the right-hand side $\underline{f}$ of the ODE system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{a})$$

symmetry is called $\mathbb{Z}_2$-symmetry because the following equivariance condition holds

$$\underline{f}(\mathbf{S}\underline{x}, \underline{a}) = \mathbf{S}\underline{f}(\underline{x}, \underline{a})$$

with $\mathbf{S}$ a matrix that obeys $\mathbf{S} \neq \mathbf{I}$ and $\mathbf{S}^2 = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix

Terminology: Kuznetsov (2004)

Equilibrium $\underline{x}^*$

One equilibrium $\underline{x}^*$:

fixed : $\mathbf{S}\underline{x}^* = \underline{x}^*$

Two equilibria $\underline{x}^*, \underline{y}^*$:

S-conjugate : if $\mathbf{S}\underline{x}^* \neq \underline{x}^*$, $\underline{y}^* = \mathbf{S}\underline{x}^*$
because $\mathbf{S}^2 = \mathbf{I}$ also $\underline{x}^* = \mathbf{S}\underline{y}^*$

Periodic solution

fixed : $\mathbf{S}\underline{x}^*(t) = \underline{x}^*(t)$

symmetric : $\mathbf{S}\underline{x}^*(t) = \underline{x}^*\left(t + \frac{T_0}{2}\right)$, period T_0

Limit cycles L

One limit cycle L:

S-invariant : $\mathbf{S}L = L$

S-invariant cycle is either fixed or symmetric

Two limit cycles L:

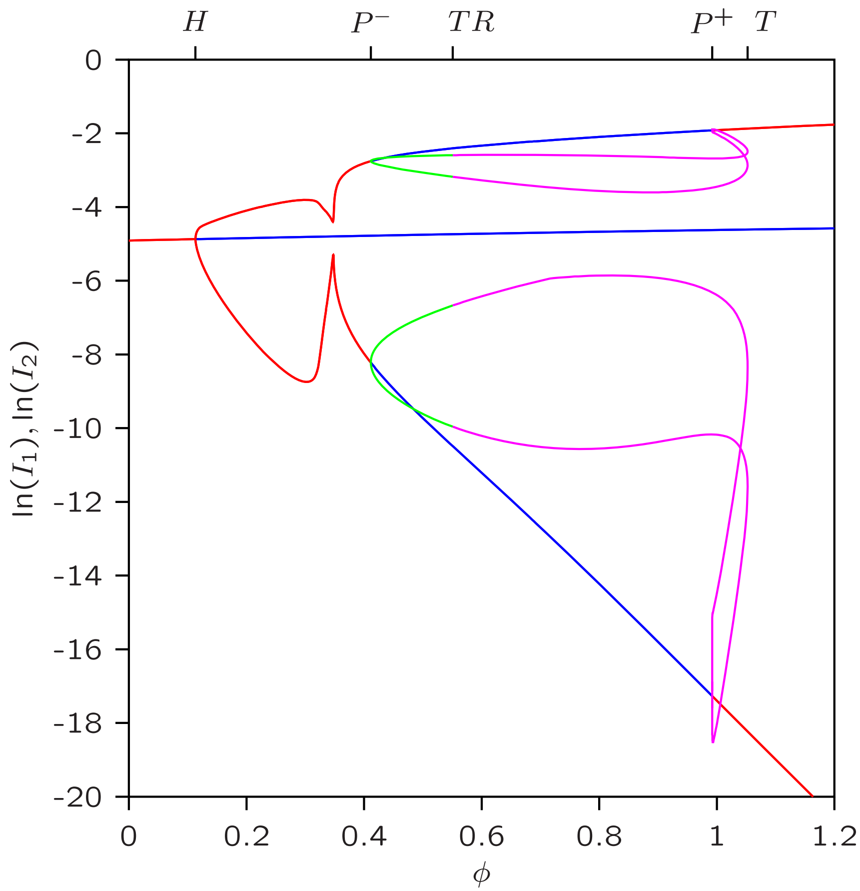
Two non-invariant limit cycles ($\mathbf{S}L \neq L$)

S-conjugate : $\underline{y}^*(t) = \mathbf{S}\underline{x}^*(t), \ \forall t \in \mathbb{R}$

The properties of symmetric systems are used with the interpretation of the numerical bifurcation analysis results for the Dengue fever model with antibody-dependent enhancement (ADE) and temporary cross immunity

Kuznetsov (2004) *Elements of Applied Bifurcation Theory* gives an overview of the possible bifurcations of equilibria and limit cycles of $\mathbb{Z}_2$-equivariant systems

One-parameter bifurcation diagram: $\alpha = 2$, ϕ free variable
I_1 and I_2



Stable Unstable Stable Unstable

The system consist of 10 ode's. For $\alpha = 2$

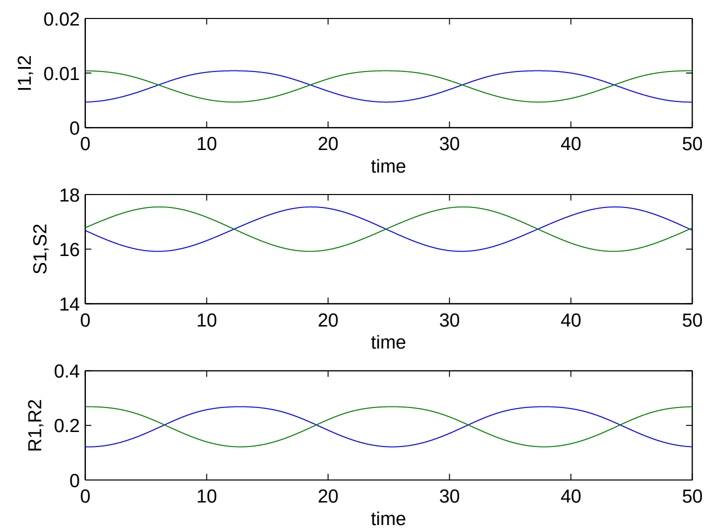
- below Hopf $\Rightarrow$ Fixed equilibrium

- Hopf H and Pitchfork P^{-} $\Rightarrow$
  Symmetric stable limit cycle

- Pitchfork $P^{-}$and Torus TR $\Rightarrow$
  Two noninvariant $\mathbf{S}$-conjugate cycles

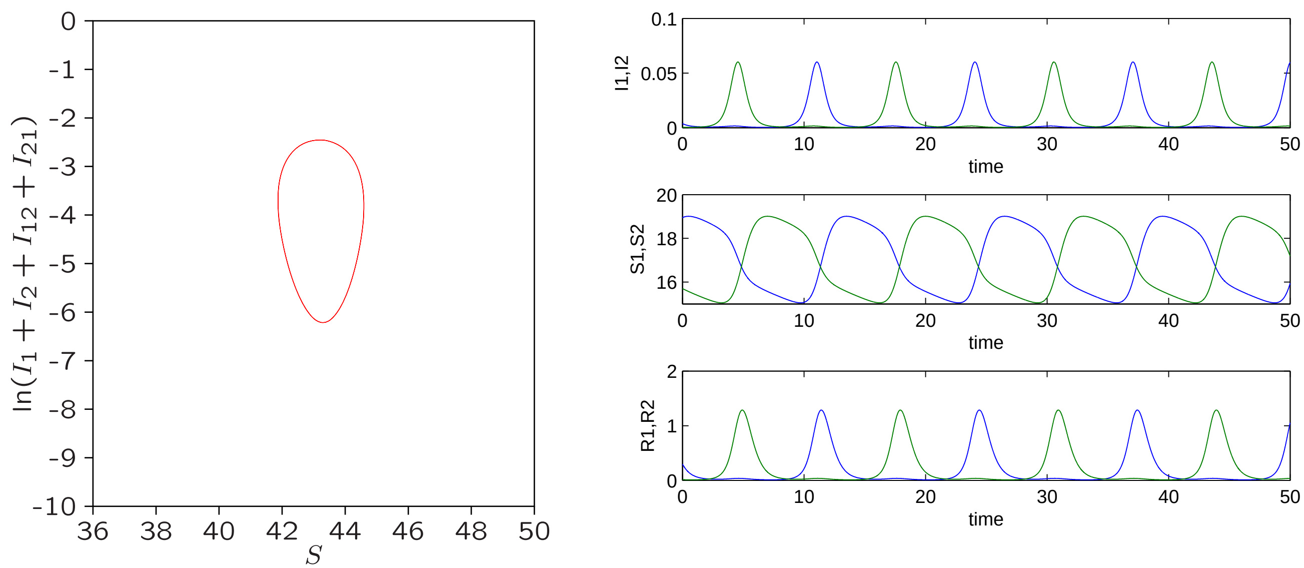
- Pitchfork P^{-} and Pitchfork P^{+} $\Rightarrow$ Chaos

limit cycles: $\phi = 0.12$, between H and P



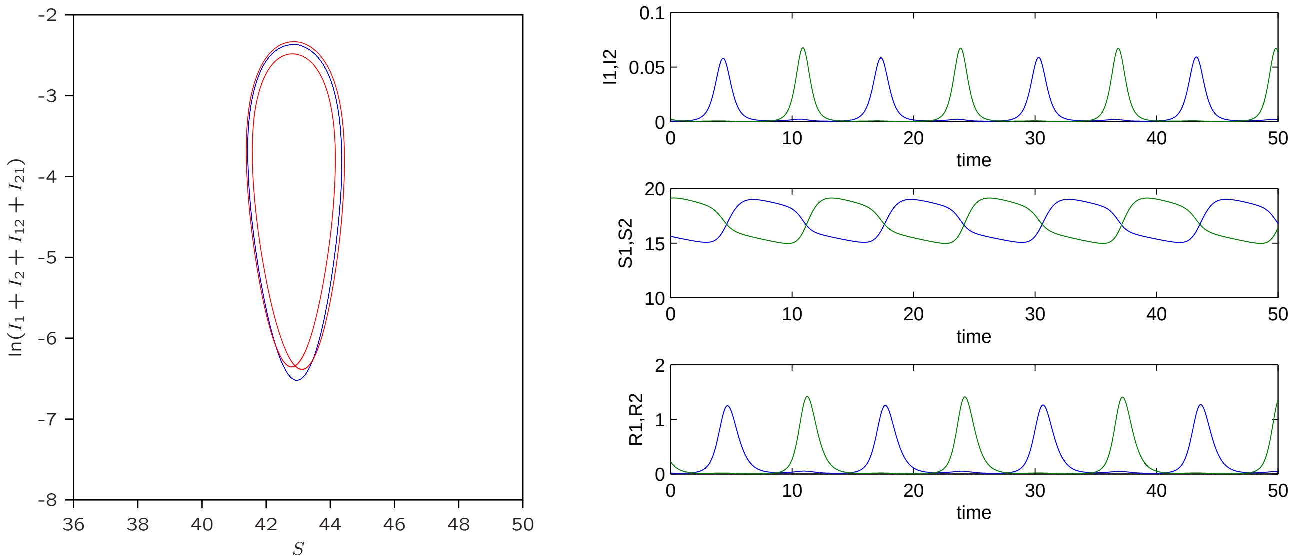
Stable symmetric $\mathbf{S}$-invariant cycle

limit cycles: $\phi = 0.4$, between H and P



Stable symmetric $\mathbf{S}$-invariant cycle

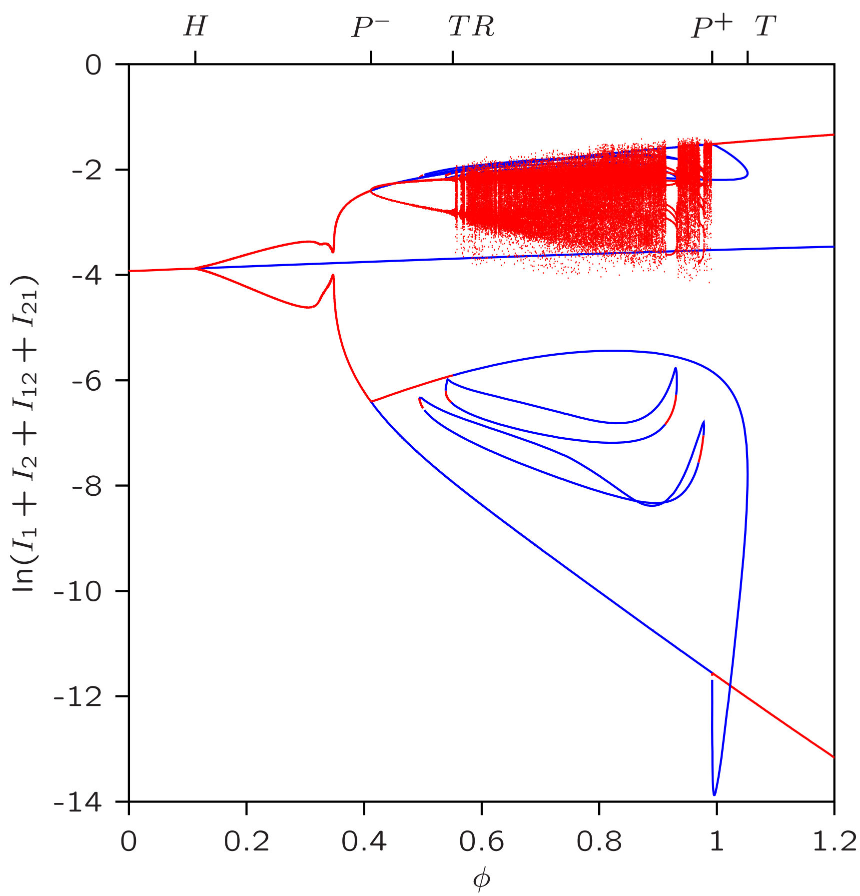
limit cycles: $\phi = 0.42$, between P and TR



Unstable symmetric $\mathbf{S}$-invariant cycle

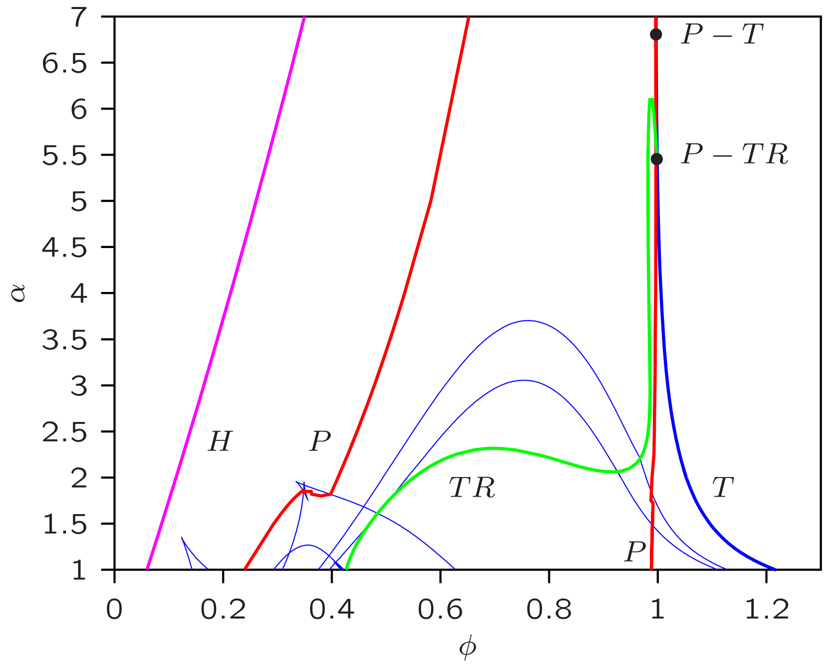
Stable Two noninvariant $\mathbf{S}$-conjugate cycles

One-parameter bifurcation diagram: total infected $I_1 + I_2 + I_{12} + I_{21}$



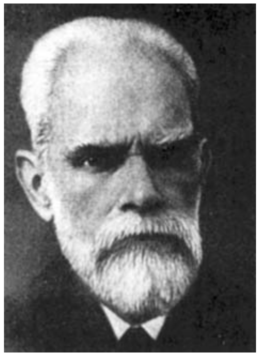
Stable Unstable

Two-parameter bifurcation diagram: α vs ϕ



Pitchfork Torus Hopf Tangent

Lyapunov exponent spectra calculations



Lyapunov exponent spectra can be used to characterize chaotic attractors

Here also to characterize the periodic solutions and the related stroboscopic maps using the definition of the Lyapunov exponents for maps

(local, one-step) Lyapunov exponent calculation

In one dimension

$$\frac{dx}{dt} = f(x)$$

Taylor series expansion

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} (x - x_0) + \cdots$$

rate of change of the distance $\Delta x(t) = (x - x_0)$ between the two trajectories

$$\frac{d\Delta x}{dt} = \frac{dx}{dt} - \frac{dx_0}{dt} = f(x) - f(x_0) = \left.\frac{df}{dx}\right|_{x_0} (x - x_0)$$

$$\Delta x(t) = \Delta x(0) e^{\lambda t} \Rightarrow \frac{d\Delta x}{dt} = \lambda \Delta x(t) = \lambda (x - x_0)$$

$$\lambda = \left.\frac{df}{dx}\right|_{x_0}$$

(global, repeated-steps) Lyapunov exponent calculation

Stability can be analysed considering small perturbations $\Delta\underline{x}(t)$ around the attractor trajectories

$$\frac{d}{dt}\Delta\underline{x} = \left.\frac{d\underline{f}}{d\underline{x}}\right|_{\underline{x}^*(t)} \cdot \Delta\underline{x}$$

Lyapunov exponents can be calculated along the trajectory as

$$\lambda_i(n) = \frac{1}{n \cdot \Delta t} \ln \left(\prod_{\nu=1}^{n} |r_{ii}(\nu)| \right)$$

where Δt is the time-step, n the (large) number of time step and r_{ii} are the diagonal elements of the upper triangular matrix R of the ν-th QR-decomposition at the ν-th time-step.

Stability and Lyapunov exponents

The set of Lyapunov exponents is called the Lyapunov spectrum which can be calculated for all parameter values.

- all Lyapunov exponents negative $\Rightarrow$ stable equilibrium
- one dominant zero Lyapunov exponent $\Rightarrow$ stable limit cycle
- two dominant zero Lyapunov exponents $\Rightarrow$ quasi-periodicity (for instance on a torus)
- a positive Lyapunov exponent $\Rightarrow$ chaotic behaviour
- multiple positive Lyapunov exponents $\Rightarrow$ hyperchaos

Limit cycles: periodic solutions with period T_0

$$\frac{d\underline{x}}{dt} = A(t)(\underline{x})$$

Matrix A Jacobian matrix is periodic with period T_0

Fundamental matrix ϕ all columns are linearly independent solutions

Principal Fundamental matrix Φ when furthermore

$\Phi(0) = \mathbf{I}$ the identity matrix

Any solution of the set of linear differential equation with periodic coefficients and initial condition $\underline{x}_0 = \underline{x}(0)$ satisfies

$$\underline{x}(T_0) = \Phi(T_0)\underline{x}_0$$

$\Phi(T_0)$ is called the Monodromy matrix

Eigenvalues μ_i of $\Phi(T_0)$ are the multipliers

There is always an eigenvalue equal to 1. The other complex multipliers determine the stability of the limit cycle

All multipliers μ_i inside unit cycle then stable and unstable otherwise

Floquet normal form

$$\phi(t) = Q(t)e^{Rt}\underline{x}_0$$

where Q and R (real) are square matrices (dimension equal to the number of differential equations)

Lyapunov exponents periodic solutions

A Floquet exponent λ_i where $e^{\lambda_i T_0}$ is a multiplier of the system

Real parts of the Floquet exponents are called Lyapunov exponents

$$\mu_i = e^{\lambda_i T_0} \Leftrightarrow \lambda_i = \frac{\ln \mu_i}{T_0}$$

The zero solution is asymptotically stable if all Lyapunov exponents are negative, Lyapunov stable if the Lyapunov exponents are non-positive and unstable otherwise.

Link Lyapunov exponents
aperiodic $\Leftrightarrow$ periodic solutions

Recall: Lyapunov exponents can be calculated along the trajectory as

$$\lambda_i(n) = \frac{1}{n \cdot \Delta t} \ln \left(\prod_{\nu=1}^{n} |r_{ii}(\nu)| \right)$$

Becomes for periodic solution where

$m \cdot \Delta t = T_0$ and $n = mN$

$$\lambda_i(mN) = \frac{1}{NT_0} \sum_{j=1}^{N} \ln \left(\prod_{\nu=1}^{m} |r_{ii}(\nu)| \right)$$
$$= \frac{1}{T_0} \ln \left(\prod_{\nu=1}^{m} |r_{ii}(\nu)| \right)$$
$$= \frac{\ln \mu_i}{T_0}$$

where μ_i is multiplier

periodic solution: Alternative methods to calculate multipliers $\Leftrightarrow$ exponents

Seasonal effects, Periodic forcing: Non-autonomous system: $\beta(t)$

$$\beta(t) = \beta_0\Big(1 + \eta\cos(\omega t)\Big)$$

with

$$\omega = 2\pi\frac{1}{T}$$

period T of forcing

$$T = 1 \;\; year$$

seasonal force is $0 \leq \eta \leq 1$

Hopf oscillator to obtain a sinusoidal signal

To use computer packages for autonomous systems such as auto, a non-autonomous system can be augmented with the following two equations

$$\begin{aligned}\dot{x} &= -\omega y + x(\eta^2 - (x^2 + y^2)) \\ \dot{y} &= \quad \omega x + y(\eta^2 - (x^2 + y^2))\end{aligned}$$

to transform non-autonomous system in an autonomous system

Polar coordinates:

$x = r \cdot \cos(\vartheta)$ and $y = r \cdot \sin(\vartheta)$
$r = \sqrt{x^2 + y^2}$ and $\vartheta = \arctan(y/x)$

The equations are now decoupled:

$$\dot{r} = r(\eta^2 - r^2)$$
$$\dot{\vartheta} = \omega$$

initial conditions $r(0) = r_0$ and $\vartheta(0) = 0$ then solution

$$r(t) = \eta \left(1 - \left(1 - \frac{\eta^2}{r_0^2}\right) e^{-2\eta^2 t}\right)^{-\frac{1}{2}}$$
$$\vartheta(t) = \omega t$$

This periodic solution is stable because we have $\lim_{t\to\infty} r(t) = \eta$, independent of $r_0 > 0$.

Stroboscopic map

We continue with the analysis of the so called Poincaré map. Since the system is periodically forced this map is also called a stroboscopic map.

The analytical expression reads with initial value r_n and $t = T = 1$

$$r_{n+1} = \eta \left(1 - \left(1 - \frac{\eta^2}{r_n^2}\right) e^{-2\eta^2}\right)^{-\frac{1}{2}}$$

where $n \in \mathbb{N}$ and initially for $n = 0$ we have $r_0 > 0$.

Asymptotically we get for large n: $r_n \to \eta$ and therefore the multiplier equals the derivative evaluated at $r = \eta$

$$\mu = \eta e^{-2\eta^2}$$

for $c = 1$ and $\eta = 1$ we get $\lambda = 0.135335$.

This single multiplier is less that 1 and therefore the periodic solution $r = \eta$ is stable

From this we can get the solution in Cartesian coordinates

$$x(t) = \eta \cdot \mathrm{cos}(2\pi t)$$
$$y(t) = \eta \cdot \mathrm{sin}(2\pi t)$$

with $0 \leq t \leq T$ where $\omega = 2\pi$ and $T = 1$

Then

$$\beta(t) = \beta_0 \Big(1 + \eta cos(\omega t)\Big)$$

becomes with $\omega = 2\pi$

$$\beta(x) = \beta_0 \Big(1 + x\Big)$$

The original 9 dimensional system is now augmented with 2 Hopf oscillator equations

$$\begin{aligned}
\dot{S} &= -\frac{\beta(x)}{N}S(I_1 + \phi I_{21}) - \frac{\beta(x)}{N}S(I_2 + \phi I_{12}) + \mu(N - S) \\
\dot{I}_1 &= \frac{\beta(x)}{N}S(I_1 + \phi I_{21}) - (\gamma + \mu)I_1 \\
\vdots &= \vdots \\
\dot{I}_{21} &= \frac{\beta(x)}{N}S_2(I_1 + \phi I_{21}) - (\gamma + \mu)I_{21} \\
\dot{x} &= -2\pi y + x(1 - (x^2 + y^2)) \\
\dot{y} &= \quad 2\pi x + y(1 - (x^2 + y^2))
\end{aligned}$$

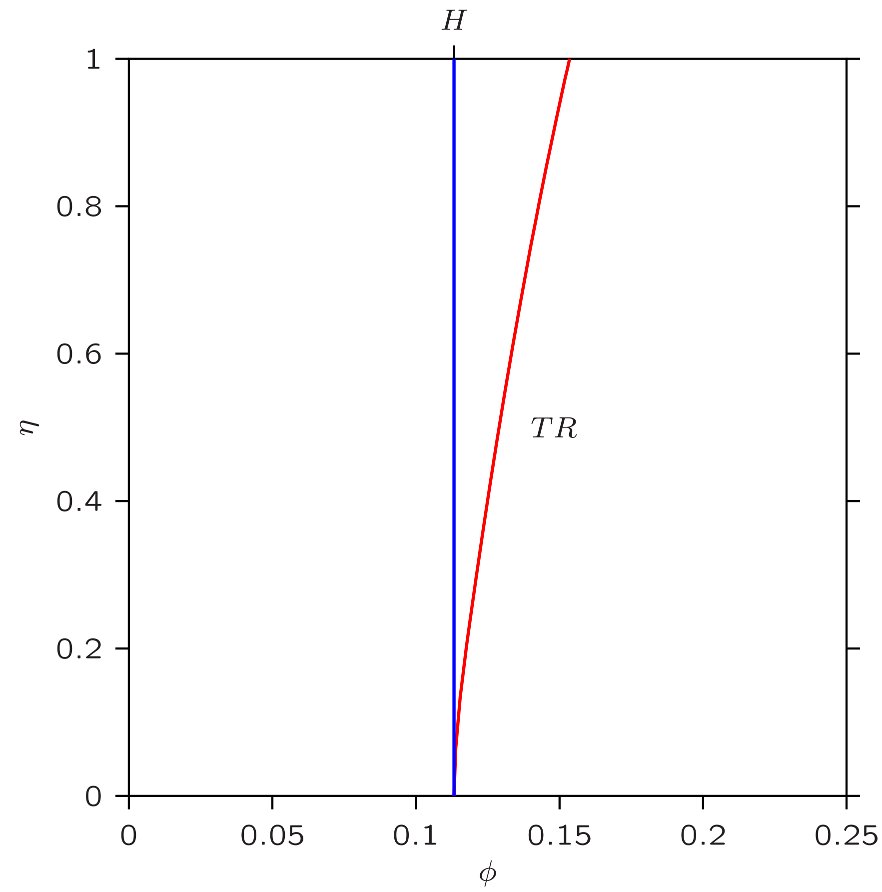
The final spectrum of 11 multipliers of this system is that of the original system plus multiplier 1 and the multiplier $\mu = 0.135335$

The characteristic equation of the monodromy matrix of the augmented system is factorized

This shows that for periodic solutions of the complete system, one multiplier equals 1 and at least one has a magnitude less than 1 while the rest of the spectrum determines its stability

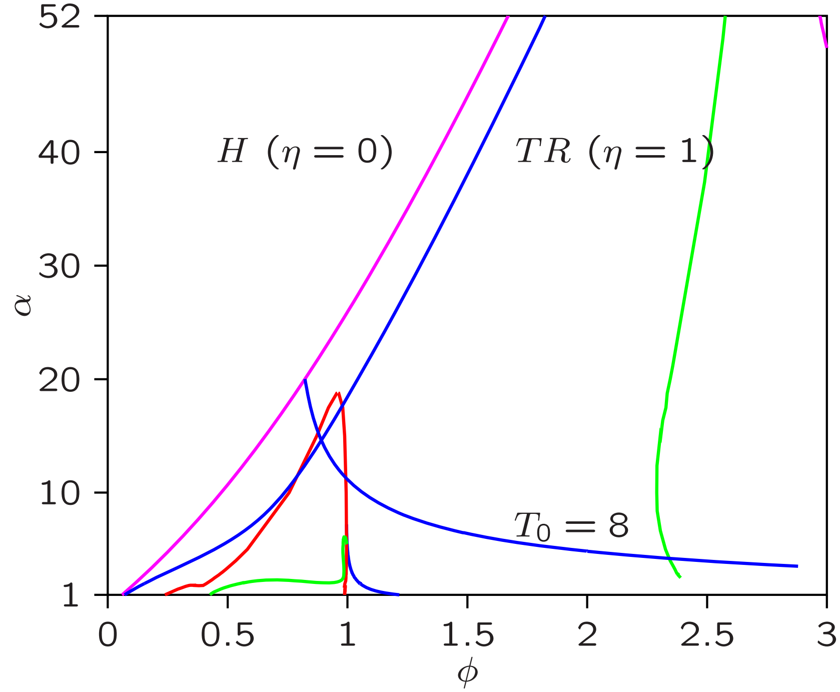
The model shows a so called $\mathbb{Z}_2$ symmetry and this leads to specific bifurcations

Two-parameter bifurcation diagram, η vs ϕ



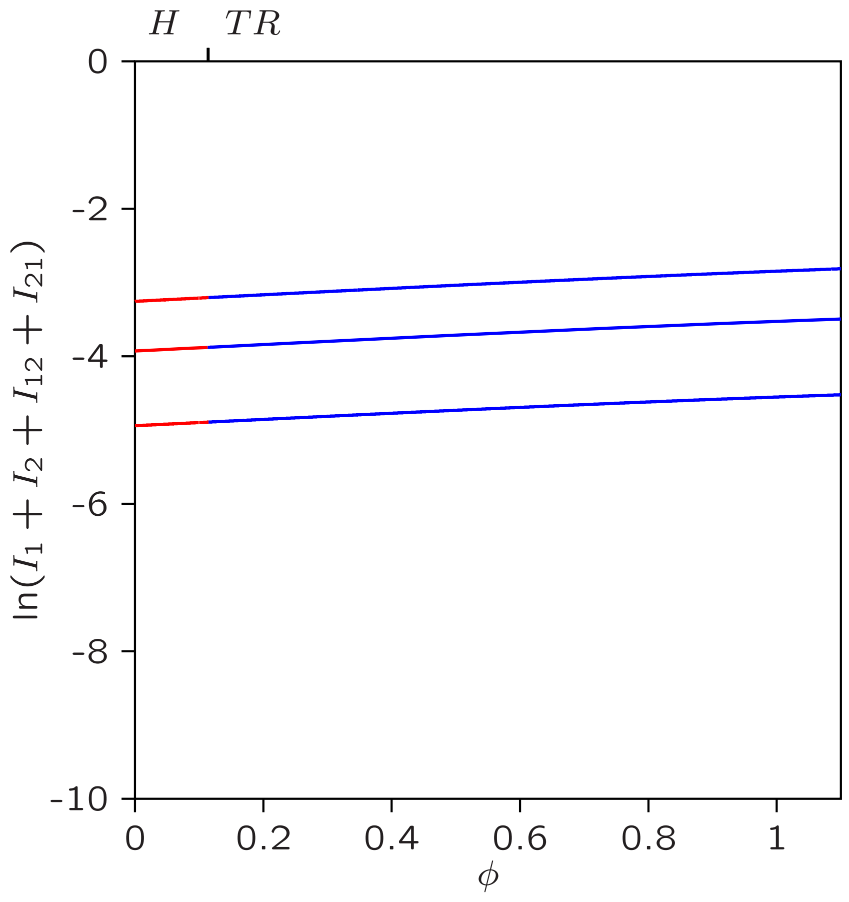
$\lim_{\eta \downarrow 0} H_{autonomous} \Rightarrow TR_{non-autonomous}$

Two-parameter bifurcation diagram, α vs ϕ



$P\ (\eta = 0)$ $TR\ (\eta = 0)$ $H\ (\eta = 0)$ $TR\ (\eta = 1)$

One-parameter bifurcation diagram, ϕ
total infected $I_1 + I_2 + I_{12} + I_{21}$



Stable Unstable

inflow of infected individuals

parameters

η: amplitude of the sinusoidal forcing function

ρ: inflow of infected individuals

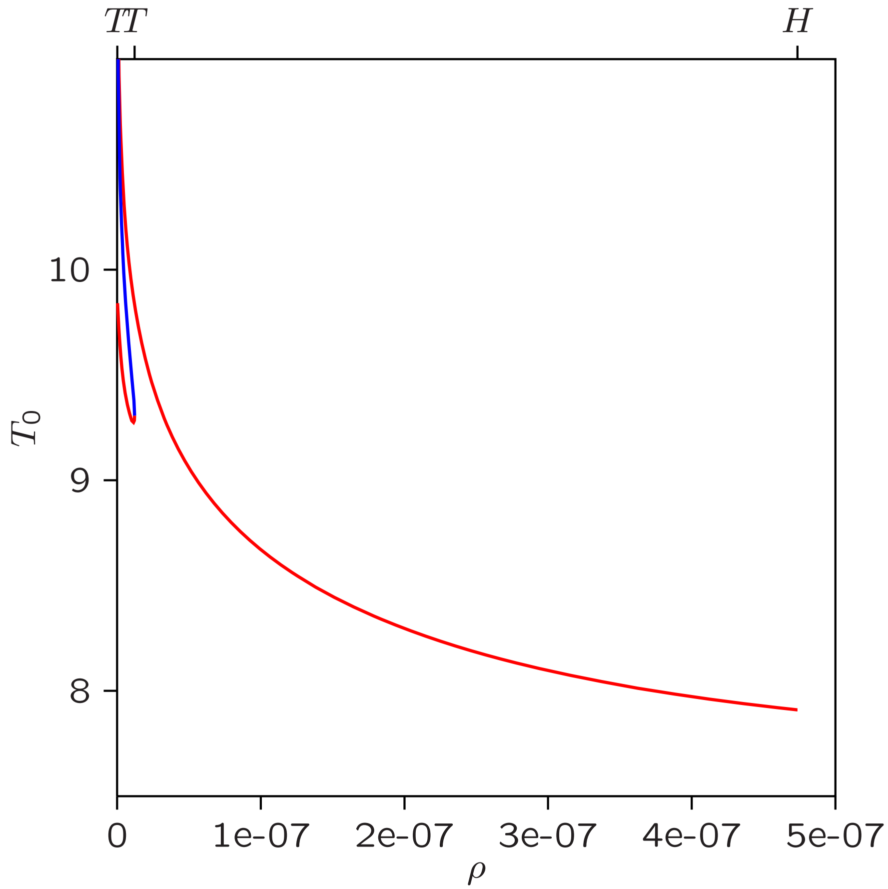
$$\dot{S} = -\frac{\beta(x)}{N} S(I_1 + \rho N + \phi I_{21}) - \frac{\beta(x)}{N} S(I_2 + \rho N + \phi I_{12}) + \mu(N - S)$$

$$\dot{I}_1 = \frac{\beta(x)}{N} S(I_1 + \rho N + \phi I_{21}) - (\gamma + \mu) I_1$$

$$\vdots = \vdots$$

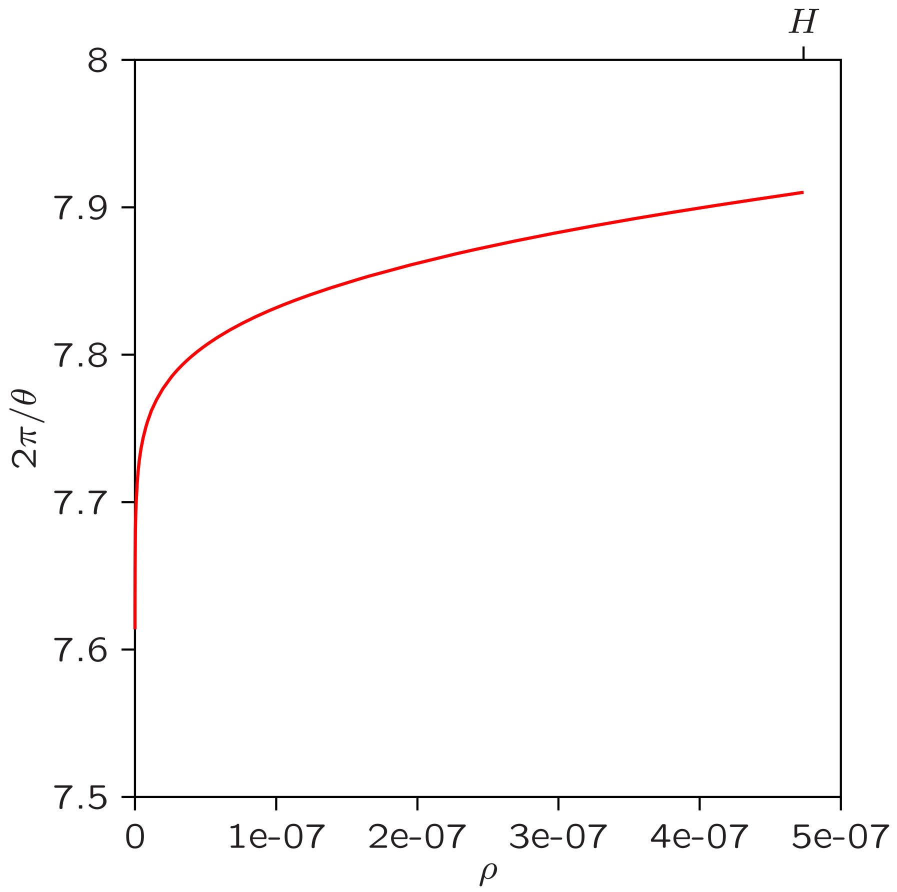
$$\dot{I}_{21} = \frac{\beta(x)}{N} S_2(I_1 + \rho N + \phi I_{21}) - (\gamma + \mu) I_{21}$$

One-parameter bifurcation diagram, T_0 period



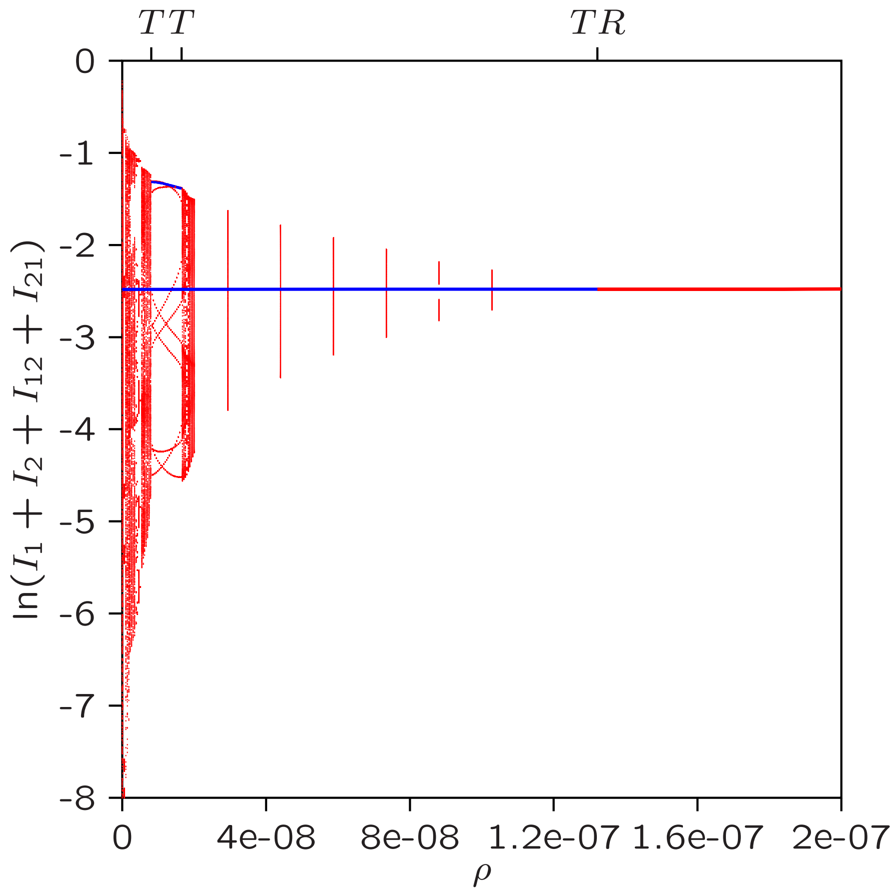
ratio of angles $2\pi/\theta$ along TR curve

θ angle between the horizontal axis and the Floquet multiplier of the complex conjugate pair with the positive imaginary part

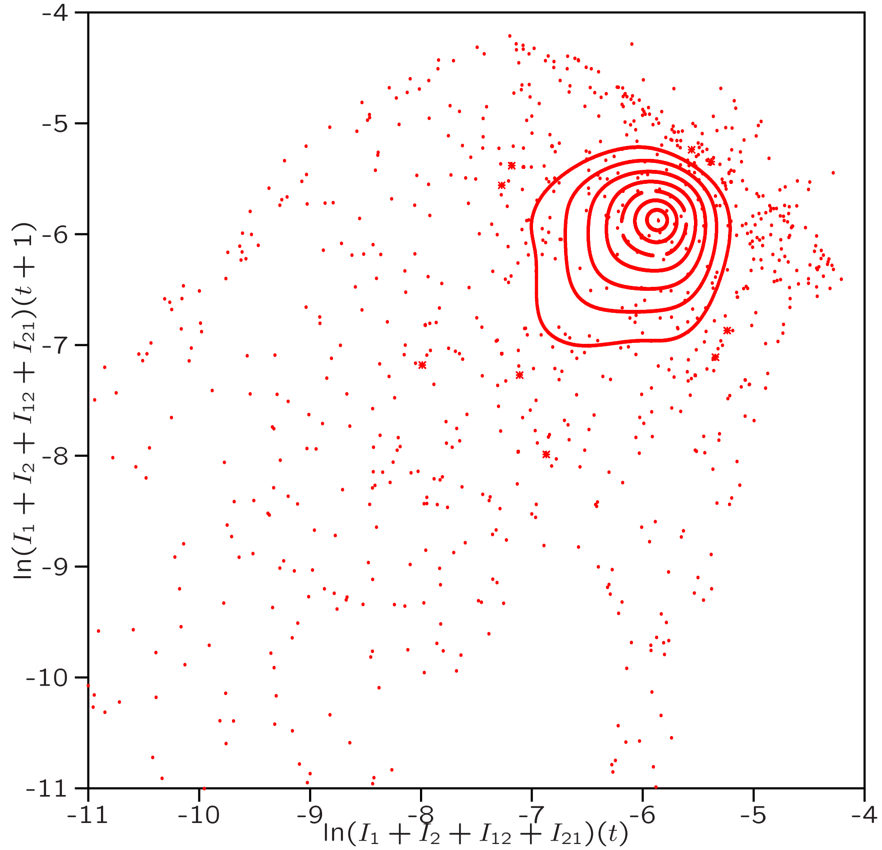


period is not multiple of forcing period

One-parameter bifurcation diagram, $\ln(I_1 + I_2 + I_{12} + I_{21})$ vs α: $\eta = 0.2$

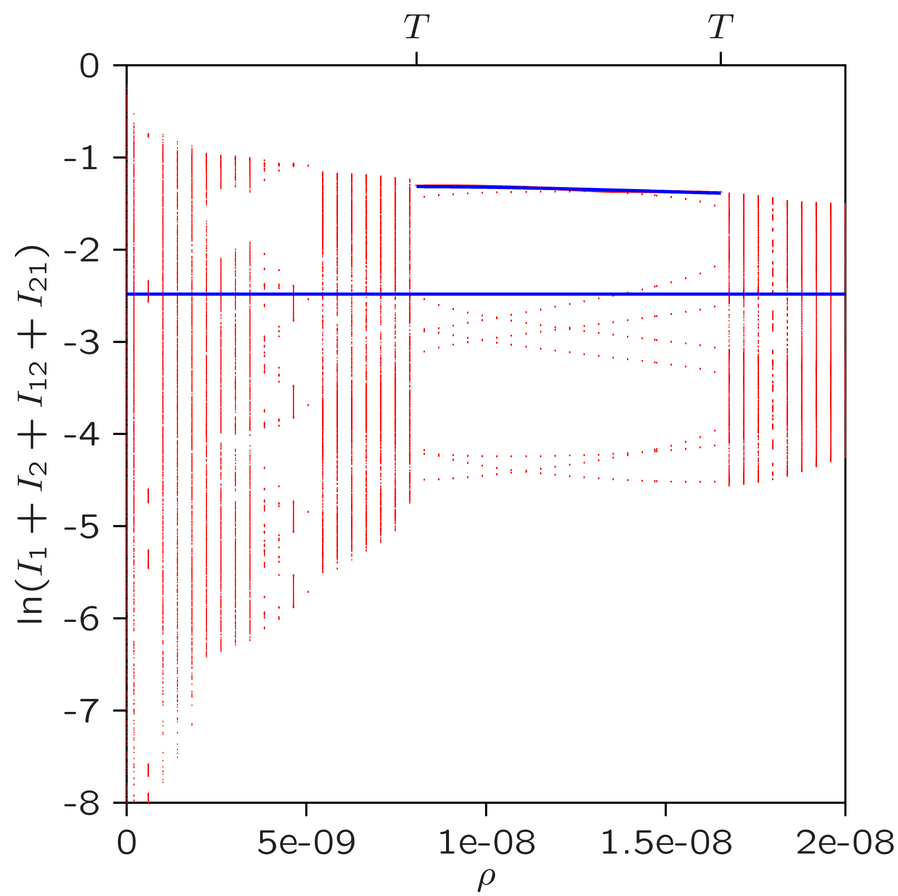


Stroboscopic map for $\ln(I_1 + I_2 + I_{12} + I_{21})$ with period of $T = 1$ year

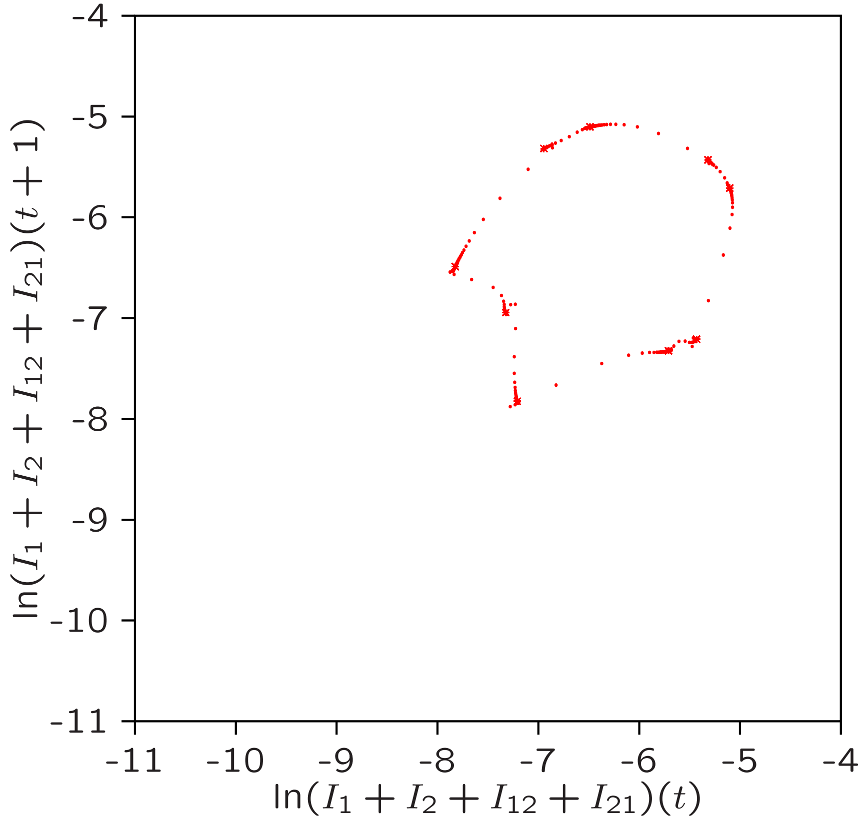


closed curves show quasi-periodic dynamics on a torus

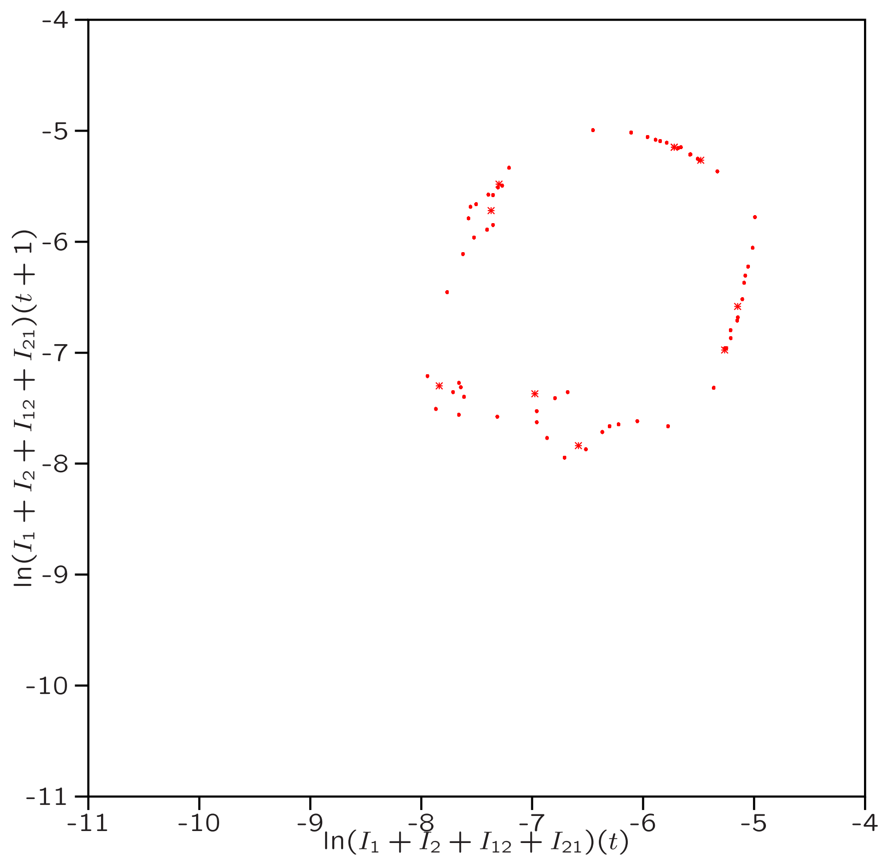
One-parameter bifurcation diagram,
$\ln(I_1 + I_2 + I_{12} + I_{21})$ vs α : $\eta = 0.2$



Stroboscopic map for $\ln(I_1 + I_2 + I_{12} + I_{21})$ with period of $T = 1$ year

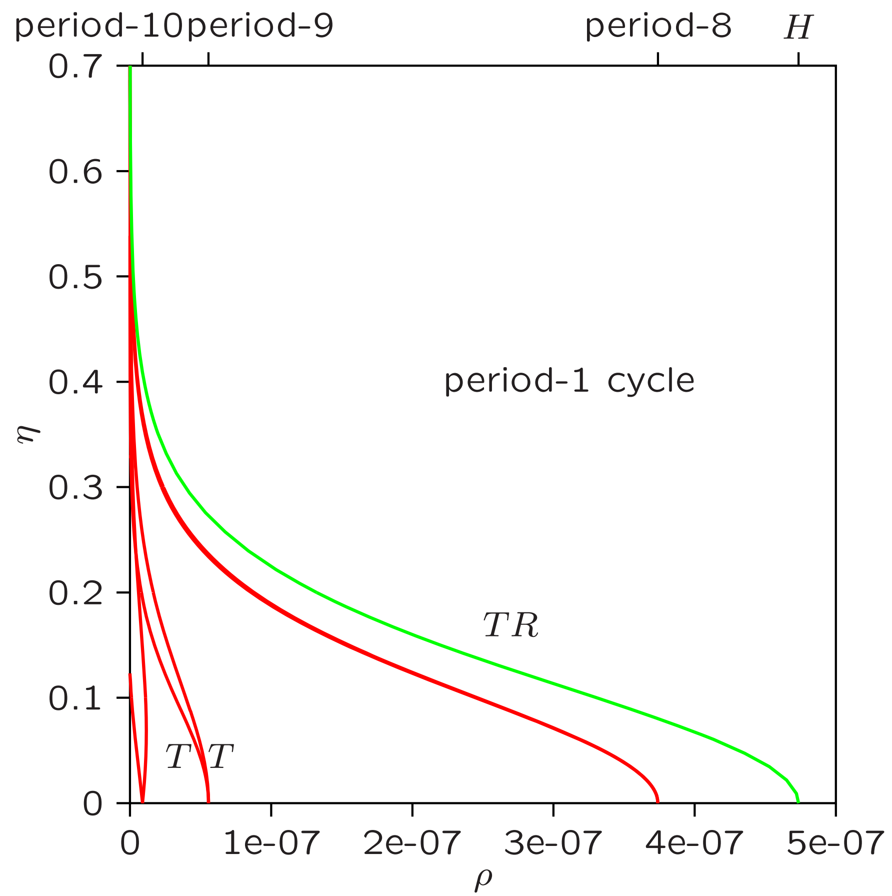


Stroboscopic map for $\ln(I_1 + I_2 + I_{12} + I_{21})$ with period of $T = 1$ year



Two-parameter bifurcation diagram, η vs ρ

Arnold tongues



Tangent Torus

Conclusions (1)

- For $\eta = 0$ the Hopf bifurcation of the autonomous system equals the Torus bifurcation of the non-autonomous system.

- In autonomous and non-autonomous system region of chaotic behaviour is bounded by a Torus bifurcation

- In non-autonomous system near the Torus bifurcation long-term periodic cycles are born and die close to rational points with respect to forcing period. So called frequency locking in Arnold tongues.

Conclusions (2)

- Bifurcation analysis is an advanced sensitivity analysis, where dependency of longterm dynamics on parameters is studied.

- only equilibria and periodic solutions can be analysed

- Lyapunov exponent calculation for chaotic dynamics

- Bifurcation analysis and Lyapunov exponent analysis are complementary

- Results are important with parameter estimation