Rank of Stably Dissipative Graphs

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Abstract

For the class of stably dissipative Lotka-Volterra systems we prove that the rank of its defining matrix is completely determined by the system's graph.

1. Introduction

Consider the system of differential equations, usually referred to as the Lotka-Volterra system,

$$\dot{x}_i(t) = x_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i(t) \geq 0$ represents the density of population i in time t and r_i its intrinsic rate of growth or decay. Each coefficient a_{ij} represents the effect of population j on population i. If $a_{ij} > 0$ this means that population i benefits from population j. $A = (a_{ij})$ is said to be the interaction matrix of the system (1).

Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ we say that system (1), or the matrix A, are *dissipative* iff there is a positive diagonal matrix D such that $Q(x) = x^T ADx \leq 0$ for every $x \in \mathbb{R}^n$.

The notion of *stably dissipative* is due to Redheffer *et al.*, who in a series of papers in the 80's [2, 3, 4, 5, 6] studied the asymptotic stability of this class of systems. Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ we say that another real matrix $\widetilde{A} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is a *perturbation of* A iff

$$\tilde{a}_{ij} = 0 \Leftrightarrow a_{ij} = 0.$$

We say that a given matrix A, or (1), is *stably dissipative* iff any sufficiently small perturbation $\widetilde{A}$ of A is dissipative, i.e.,

$$\exists\, \epsilon > 0 : \max_{i,j} |a_{ij} - \tilde{a}_{ij}| < \epsilon \;\Rightarrow\; \widetilde{A} \text{ is dissipative.}$$

When A is dissipative, the system (1) with equilibrium point $q \in \mathbb{R}^n$ admits the Lyapunov function

$$h(x) = \sum_{i=1}^{n} \frac{x_i - q_i \log x_i}{d_i}, \tag{2}$$

and by La Salle's theorem [1] we know that the attractor is contained in the set $\{\, \dot{h} = 0 \,\}$.

From the interaction matrix A we can construct a graph G_A having as vertices the n species $\{1, \ldots, n\}$. An edge is drawn connecting the vertices i and j whenever $a_{ij} \neq 0$ or $a_{ji} \neq 0$. Redheffer *et al.* [2, 3, 4, 5, 6] have characterized the class of stably dissipative systems and its attractor in terms of the graph G_A. In particular, they describe a simple reduction algorithm, running on the graph G_A, that holds for every stably dissipative system with interaction graph G_A.

To describe their algorithm they convention that a vertex i is coloured black, $\bullet$, if one can prove that $x_i = q_i$ on the attractor, a cross is drawn at a vertex i, $\oplus$, if one can prove that x_i is constant on the attractor, and finally, all other vertices are coloured white, $\circ$.

To start this algorithm, as $a_{ii} \leq 0$ for all i, they

(I) colour in black, $\bullet$, every vertex $i \in \{1, \ldots, n\}$ such that $a_{ii} < 0$, which implies that $x_i = q_i$ on the attractor, and in white, $\circ$, all other vertices.

The reduction procedure consists of the following rules, corresponding to valid inference rules:

(a) If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$, except for one vertex l, then colour l as $\bullet$;

(b) If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$ or $\oplus$, except for one vertex l, then draw $\oplus$ at the vertex l;

(c) If j is a $\circ$-vertex and all of is neighbours are $\bullet$ or $\oplus$, then draw $\oplus$ at the vertex j.

Redheffer *et al.* define the *reduced graph* of the system, $\mathcal{R}(G_A)$, as the graph obtained from G_A by successive applications of the reduction rules (a), (b) and (c) until they can no longer be applied. In [4] Redheffer and Walter proved the following result, which in a sense states that the previous algorithm on G_A can not be improved.

Theorem. *Given a stably dissipative matrix A,*

(a) If $\mathcal{R}(G_A)$ has only $\bullet$-vertices then A is nonsingular, the stationary point q is unique and every solution of (1) converges, as $t \to \infty$, to q.

(b) If $\mathcal{R}(G_A)$ has only $\bullet$ and $\oplus$-vertices, but not all $\bullet$, then A is singular, the stationary point q is not unique, and every solution of (1) has a limit, as $t \to \infty$, that depends on the initial condition.

(c) If $\mathcal{R}(G_A)$ has at least one $\circ$-vertex then there exists a stably dissipative matrix $\widetilde{A}$, with $G_{\widetilde{A}} = G_A$, such that the system (1) associated with $\widetilde{A}$ has a nonconstant periodic solution.

Consider now the stably dissipative matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & -1 & 2 \\ 0 & 0 & -2 & 0 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ -1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Drawing the graphs G_A and G_B, we see that they are equal to the graph in fig. 1.

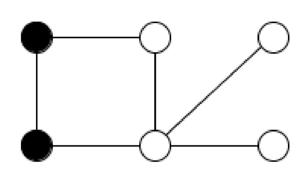

Figure 1: *The graph $G_A = G_B$.*

If we calculate the rank of the matrices A and B we see that $\mathrm{rank}(A) = \mathrm{rank}(B) = 5$. Is that a coincidence, or there's any relation with the fact that both matrices share the same graph?

Recently, Zhao and Luo [8] gave a complete characterization of stably dissipative matrices.

Theorem. *Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, A is stably dissipative iff every cycle of G_A contains at least a strong link, i.e., an edge between $\bullet$-vertices ($\bullet$–$\bullet$) and there is a positive diagonal matrix D such that AD is almost skew-symmetric.*

2. Main Results

Theorem. *Given a graph G, all stably dissipative matrix A with graph G have the same rank.*

By this theorem we can define the *rank* of a stably dissipative graph G, denoted hereafter by $\mathrm{rank}(G)$, as the rank of any stably dissipative matrix with graph G.

Given a stably dissipative graph G, i.e., the graph of a stably dissipative matrix, if G has no extreme $\circ$-vertex then $\mathcal{R}(G)$ has only $\bullet$-vertices, and in this case we can apply (a) of the first theorem. Otherwise, if G has some extreme $\circ$-vertex i, we define the *trimmed* graph $T_i(G)$ as follows:

Starting from **(I)**, let the vertex i' be the unique connected to i by some edge of G. Then $T_i(G)$ is the partial graph obtained from G by removing every other edge incident with i'.

Figure 2: *A graph G and its trimmed $T_i(G)$.*

These operations in the graph correspond to annihilate some entries of the associated matrix, however, the matrix rank does not change, as we state in the theorem below.

We say that a graph G has *constant rank* if every stably dissipative matrix A with graph G has the same rank.

Theorem (Trimming theorem)**.** *Let i be some extreme $\circ$-vertex of a graph G of a stably dissipative matrix. If $T_i(G)$ has constant rank then so has G, and $\mathrm{rank}(G) = \mathrm{rank}(T_i(G))$.*

The Trimming theorem gives a simple recipe to compute the rank of a graph:

(R) Trim G while possible. In the end, discard the single $\circ$-vertex components and count the remaining vertices.

Now back to our example we can see in Fig. 3 the original graph and its trimmed. Applying **(R)** we can conclude, as we stated before, that all stably dissipative matrices that share this graph have rank 5.

Figure 3: *The graph $G_A = G_B$ and its trimmed.*

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